Velocity-Field Theory, Boltzmann's Transport Equation, Geometry and Emergent Time

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Abstract

Boltzmann equation describes the time development of the velocity distribution in the continuum fluid matter. We formulate the equation using the field theory where the *velocity-field* plays the central role. The matter (constituent particles) fields appear as the density and the viscosity. *Fluctuation* is examined, and is clearly discriminated from the quantum effect. The time variable is *emergently* introduced through the computational process step. The collision term, for the (velocity)**4 potential (4-body interaction), is explicitly obtained and the (statistical) fluctuation is closely explained. The present field theory model does *not* conserve energy and is an open-system model. (One dimensional) Navier-Stokes equation , Burger's equation, appears. In the latter part, we present a way to directly define the distribution function by use of the geometry, appearing in the mechanical dynamics, and the Feynman's path-integral.

Key Words : Boltzmann equation; velocity field theory; statistical fluctuation; computational step number; open system; geometry.

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1 Introduction

Boltzmann equation was introduced to explain the second law of the thermodynamics in the dynamical way, in 1872, by Boltzmann. We considers the (visco-elastic) fluid matter and examine the dynamical behavior using the velocity-field theory. The scale size we consider is far bigger than the atomic scale ($\sim 10^{-10}$m) and is smaller than or nearly equal to the optical microscope scale ($\sim 10^{-6}$m). The equation describes the temporal development of the distribution function $f(t, \mathbf{x}, \mathbf{v})$ which shows the probability of a fluid-molecule (particle) having the velocity $\mathbf{v}$ at the space $\mathbf{x}$ and time t.

We reformulate the Boltzmann equation using the field theory of the velocity field $\mathbf{u}(\mathbf{x},\ 't')$. Basically it is based on the *minimal energy principle*. We do *not* introduce time t. Instead of t, we use the *computational step number* n. The system we consider consists of the huge number of fluid-particles (molecules) and the physical quantities, such as energy and entropy, are the *statistically-averaged* ones. It is not obtained by the deterministic way like the classical (Newton) mechanics. We introduce the *statistical ensemble* by using the well-established field-theory method, the *background-field method*[5, 6]. Renormalization phenomenon occurs not because the quantum effect but because the statistical fluctuation due to the inevitable uncertainty caused by 1) step-wise (discrete-time) formulation and 2) continuum formulation for the space.

The dissipative system we consider is characterized by the dissipation of energy. Even for the particle classical (Newton) mechanics, the notion of energy is obscure when the dissipation occurs. We consider the movement of a particle under the influence of the friction force. The emergent heat (energy) during the period [t_1, t_2] can *not* be written as the following popularly-known form.

$$\int_{x_1}^{x_2} F_{\text{friction}}\, dx = [E(x(t), \dot{x}(t))]_{t_1}^{t_2} = E|_{t_2} - E|_{t_1}\ ,\ x_1 = x(t_1)\ ,\ x_2 = x(t_2)\ , \quad (1)$$

where $x(t)$ is the orbit (path) of the particle. It depends on the path (or orbit) itself. It cannot be written as the form of difference between some quantity at time t_1 and t_2. In this situation, we realize the time itself should be reconsidered when the dissipation occurs. Due to Einstein's idea of "space-time democracy", we have stuck to the standpoint that space and time should be treated on the equal footing. We present here the step-wise approach to the time-development.

We do *not* use time variable. Instead we use the computational-process step number n. Hence the *increasing* of the number n is identified as the *time development*. The connection between step n and step $n-1$ is determined

by the *minimal energy principle*. In this sense, time is "emergent" from the minimal energy principle. The direction of flow (arrow of time) is built in from the beginning. [1]

In the latter part of this paper, an approach to the statistical averaging procedure , based on the geometry of the mechanical dynamics, is presented.

The content is described as follows. The step-wise dynamical equation is presented in Sec.2. We start with the n-th step energy functional. By regarding n steps as the time t_n, we derive (1 dim) Navier-Stokes equation. In Sec.3, the orbit (path) of the fluid particle is explained in this step-wise formalism. The total energy and the energy rate are also explained. The statistical fluctuation is closely explained in Sec.4. Especially the difference from the quantum effect is stressed. Using the path-integral, we take into account the fluctuation effect. Owing to the present velocity-field formalism, we can obtain Boltzmann's equation, as described in Sec.5, up to the collision term. This step-wise approach is applied to the mechanical system in Sec.6. We take a simple dissipative model: the harmonic oscillator with friction. The trajectory is solved in the step-wise way. We find the total energy changes as the step proceeds. From the n-step energy expression we can extract the geometrical structure (the metric) of the trajectory. The metric is used, in Sec.7, to define the statistical ensemble of the system of N viscous particles. We propose some models using the geometrically-basic quantities: the length and the area. Conclusion is given in Sec.8. A few appendices are provided to supplement the text. App.A treats (1+3) dimensional field theory in this step-wise formalism. App.B is the calculation of the statistical fluctuation effect and it supplements Sec.4. An additional mechanical model (Spring-Block model) is described with the calculation result in App.C.

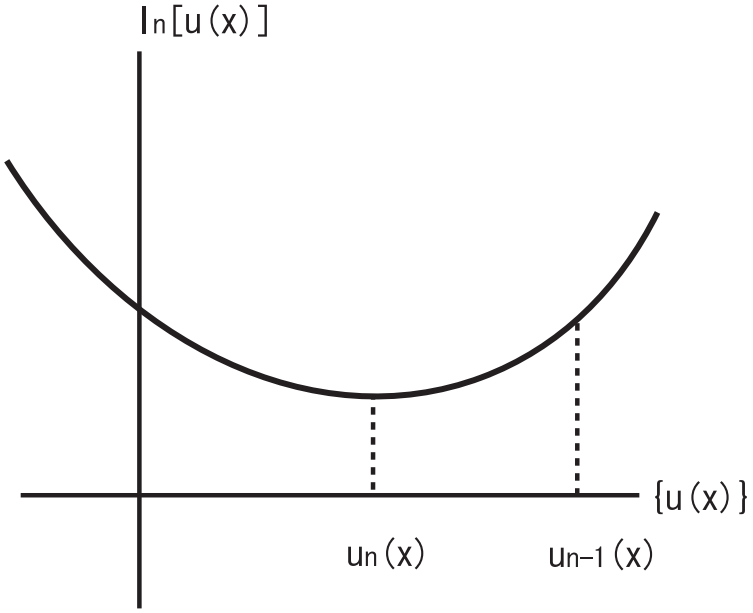
2 Emergent Time and Diffusion (Heat) Equation

We consider 1 dimensional viscous fluid , and the velocity field $\{u(x); -\infty < x < \infty\}$ describes the velocity distribution in the 1 dim space. Let us take the following energy functional[2, 3] of the velocity-field u(x),

$$I_n[u(x)] = \int dx\{\frac{\sigma}{2\tilde{\rho}_0}(\frac{du}{dx})^2 + V(u) + u\frac{dV^1(x)}{dx} + \frac{1}{2h}(u - u_{n-1})^2\} + I_n^0 \quad ,$$
$$\sigma \equiv 1 \quad , \quad \tilde{\rho}_0 \equiv 1 \quad , \qquad n = 1, 2, \cdots \quad ,$$

[1] For a recent review on the nature of time, see ref.[1].

Figure 1: The energy functional $I_n[u(x)]$, (2), of the velocity-field $u(x)$.



$$V(u) = \frac{m^2}{2}u^2 + \frac{\lambda}{4!}u^4 \quad , \quad u = u(x) \ , \ u_{n-1} = u_{n-1}(x) \quad , \ (2)$$

where $\tilde{\rho}_0$ is the constant *mass-density* and is taken to be 1. [2] I_n^0 is a 'constant' term which is independent of $u(x)$. Later we will fix it. σ is the *viscosity* constant and is also taken to be 1. [3] m^2 is the mass density: (the mass of the fluid-particle)/ $2l$. The quantity (2) is the total energy of the fluid. The *velocity potential* $V(u)$ has the mass term and the 4-body interaction term. $V^1(x)$ is the (ordinary) position-dependent potential. $\frac{dV^1(x)}{dx}$ is the external

[2] The time-development term $\mathcal{W}_n(u)$ is generally written as

$$I_n[u(x)] = \int dx\{\frac{\sigma}{2\tilde{\rho}_0}(\frac{du}{dx})^2 + V(u) + u\frac{dV^1(x)}{dx} + \mathcal{W}_n(u)\} + I_n^0 \quad ,$$
$$\text{where} \quad \mathcal{W}_n(u) = \frac{1}{2h}(u - u_{n-1})^2 \quad \text{or} \quad \frac{1}{2h^2}(u - 2u_{n-1} + u_{n-2})^2 \quad . \qquad (3)$$

The latter case can be treated in the same way and is given in App.A. Besides the velocity field $u(x)$, the fluid matter density field $\tilde{\rho}(x)$ generally appears. We here consider the incompressible case: $\tilde{\rho}(x) = \tilde{\rho}_0(\text{constant}) \equiv 1$.

We list the physical dimensions of various quantities. $[x]$=$[l]$=L, $[u]$=L/T, $[h]$=L/M, $[m^2]$=M/L, $[\lambda]$=$\mathrm{MT}^2/\mathrm{L}^3$, $[\sigma]$=M^2, $[\tilde{\rho}]$= $[\tilde{\rho}_0]$=M/L, $[I_n]$=$\mathrm{ML}^2\mathrm{T}^{-2}$, $[V]$=MLT^{-2}. Furthermore we have, $[\sqrt{\lambda\sigma}]$=$(\mathrm{M/L})^{3/2}\mathrm{T}$, $[\sqrt{h^3\lambda\sigma}]$= $[m^{-3}\sqrt{\lambda\sigma}]$=$[h\sqrt{\lambda\sigma}/m]$=T.

[3] We often keep writing σ and $\tilde{\rho}_0$ in order to see the dimensional consistency.

source (force) in this velocity-field theory. h is some constant which will be identified as the *time-separation* for one step. $u_{n-1}(x)$ is taken to be a *given* velocity field at the (n-1)-th step. The n-th step velocity field $u_n(x)$ is given by the *minimal principle* of the n-th energy functional $I_n(u)$. This approach is callled "discrete Morse flows method"[2, 3, 4].

For simplicity we take the *periodic* boundary condition for the space.

$$u(x) = u(x + 2l) \quad , \tag{4}$$

where $2l$ is the periodic length. We may restrict the space region as $-l \leq x \leq l$. The variation equation $\delta I_n(u) = 0(u(x) \to u(x) + \delta u(x))$ gives

$$\frac{1}{h}(u_n(x) - u_{n-1}(x)) = \frac{\sigma}{\tilde{\rho}_0}\frac{d^2u_n}{dx^2} - \frac{\delta V(u_n)}{\delta u_n} - \frac{dV_n^1(x)}{dx} \quad ,$$
$$\frac{\delta V(u)}{\delta u} = m^2u + \frac{\lambda}{3!}u^3 \quad , \tag{5}$$

where we have replaced the *minimal solution* by u_n. From the construction, we have the relation:

$$I_n[u_n] \quad \leq \quad I_n[u_{n-1}] \quad . \tag{6}$$

We, however, *cannot* say $I_n[u_n] \leq I_{n-1}[u_{n-1}]$. The above equation describes the n-th step velocity field $u_n(x)$ in terms of $u_{n-1}(x)$ and vice versa. Hence it can be used for the *computer simulation*.

We here introduce the *discrete time* variable t_n as the step number n of u_n.

$$t_n = nh = n\tau_0 \times (\frac{h}{\tau_0}) \quad , \quad \tau_0 \equiv h\sqrt{\lambda\sigma}/m \quad , \quad n = 1,\ 2,\ \cdots$$
$$t_0 \equiv 0 \quad , \tag{7}$$

where τ_0 is the time unit. [4] The eq.(5) is, in terms of the 'renewed' field $u(x,t)$, expressed as

$$\frac{1}{h}(u(x, t_{n-1} + h) - u(x, t_{n-1})) =$$
$$\frac{\partial^2 u(x,t_n)}{\partial x^2} - \frac{\delta V(u(x,t_n))}{\delta u(x,t_n)} - \frac{\partial V^1(x,t_n)}{\partial x} \quad , \tag{8}$$

[4] Note $[\tau_0]$=T. See the footnote of eq.(2). Generally the time t_n can be introduced by $t_n = f(n)h$ where $f(n)$ is a function of n. The form of $f(n)$ defines the time coordinate. The change of the form is the transformation of the time coordinate. The simple one is $f(n) = an + b$. In the text $f(n) = n$ is taken. If we take $f(n) = -n$, the time flow is introduced in the inverse way.

where we use $u(x,t_n) \equiv u_n(x),\ t_n = t_{n-1} + h$. As $h \to 0$, we obtain

$$\frac{\partial u(x,t)}{\partial t} = \frac{\sigma}{\tilde{\rho}_0}\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\delta V(u(x,t))}{\delta u(x,t)} - \frac{\partial V^1(x,t)}{\partial x} \quad , \tag{9}$$

where we have replaced both t_n and t_{n-1} by t. This is 1 dim *diffusion* equation with the potential $V(u)$.

We remind that the variational principle for the n-step energy functional $I_n[u(x)]$ (2), $\delta I_n = I_n[u+\delta u] - I_n[u] = 0$, gives $u_n(x)$ for the given $u_{n-1}(x)$. We regard the increase of the step number as the *time development*. [5] Taking into account the fact that, at the (n-1)-step, the matter-particle at the point x flows at the speed of $u_{n-1}(x)$, the energy functional I_n, (2), should be replaced by the following one[2, 3].

$$\tilde{I}_n[u(x)] = \int dx\{\frac{\sigma}{2\tilde{\rho}_0}(\frac{du}{dx})^2 + V(u) + u\frac{dV^1(x)}{dx} + \frac{1}{2h}(u(x+hu_{n-1}) - u_{n-1})^2\} + \tilde{I}_n^0 \quad ,$$
$$\sigma \equiv 1 \quad , \quad \tilde{\rho}_0 \equiv 1 \quad , \qquad n = 1, 2, \cdots \quad ,$$
$$V(u) = \frac{m^2}{2}u^2 + \frac{\lambda}{4!}u^4 \quad , \quad u = u(x)\ ,\ u_{n-1} = u_{n-1}(x) \quad . \tag{10}$$

Note that $u(x) - u_{n-1}(x)$ in eq.(2) is replaced by $u(x + hu_{n-1}(x)) - u_{n-1}(x)$. For the simple case of *no potential*, $V = 0$, and no external force, $\frac{dV^1}{dx} = 0$,

$$J_n[u(x)] = \int dx\{\frac{1}{2}(\frac{du}{dx})^2 + \frac{1}{2h}(u(x + h\ u_{n-1}(x)) - u_{n-1}(x))^2\} +' \text{const}' \quad , \tag{11}$$

The above functional is equivalent to $I_n[u(x)]$ with the potential.

$$V(u) = (u(x) - u_{n-1}(x))u_{n-1}(x)\frac{du(x)}{dx} + O(h) \quad . \tag{12}$$

where we consider the case of sufficiently-small h. Eq.(5) gives us the following equation as the minimal equation for $J_n[u] :\ \delta J_n[u] = 0$

$$\frac{1}{h}(u_n(x) - u_{n-1}(x)) = \frac{d^2u_n}{dx^2} - u_{n-1}(x)\frac{du_n(x)}{dx} + O(h) \quad . \tag{13}$$

Hence the step-wise recursion relation (5) is corrected as

$$\frac{1}{h}(u_n(x) - u_{n-1}(x)) + u_{n-1}(x)\frac{du_n(x)}{dx} = \frac{\sigma}{\tilde{\rho}_0}\frac{d^2u_n}{dx^2} - \frac{\delta V(u_n)}{\delta u_n} - \frac{dV_n^1(x)}{dx} \quad , \tag{14}$$

[5] Time is defined here by the energy-minimal principle.

As done before, let us replace the step number n by the *discrete* time $t_n = nh$. Taking the continuous time limit ($h \to 0$), we obtain

$$\frac{\partial u(x,t)}{\partial t} + u(x,t)\frac{\partial u(x,t)}{\partial x} = \frac{\sigma}{\tilde{\rho}_0}\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\delta V(u(x,t))}{\delta u(x,t)} - \frac{\partial V^1(x,t)}{\partial x} \quad , \quad (15)$$

This is called *Burgers's equation* (with the velocity potential $V(u)$ and the external force $\frac{\partial V^1}{\partial x}$) and is considered to be 1 dimensional Navier-Stokes equation. Note that the non-linear term in the LHS of eq.(15) appears not from the potential (velocity-field interaction) but from the *self-consistency* of the velocity-field change from step $n-1$ to step n. $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{Du}{Dt}$ is called *Lagrange derivative*.

The equation (15), for the massless case $m = 0$, is invariant under the *global Weyl transformation*.

$$\begin{aligned} V^1(x,t) \quad &\to \quad \mathrm{e}^{-2\varepsilon} V^1(\mathrm{e}^{\varepsilon}x, \mathrm{e}^{2\varepsilon}t) \quad , \\ u(x,t) \quad &\to \quad \mathrm{e}^{-\varepsilon} u(\mathrm{e}^{\varepsilon}x, \mathrm{e}^{2\varepsilon}t) \quad , \\ \frac{\partial}{\partial x} = \partial_x \quad &\to \quad \mathrm{e}^{-\varepsilon}\partial_x \quad , \quad \frac{\partial}{\partial t} = \partial_t \quad \to \quad \mathrm{e}^{-2\varepsilon}\partial_t \quad , \\ t \quad &\to \quad \mathrm{e}^{2\varepsilon}t \quad , \quad x \quad \to \quad \mathrm{e}^{\varepsilon}x \quad , \end{aligned} \quad (16)$$

where ε is the real constant parameter. [6] When it is small : $|\varepsilon| \ll 1$, the variation δu is given by

$$\delta u = \varepsilon\{-u + x\ \partial_x u + 2t\ \partial_t u\} + O(\varepsilon^2) \quad , \quad (17)$$

For simplicity, we explain in one space-dimension (dim). The generalization to 2 dim and 3 dim is straightforward. Furthermore the ordinary field theory (not using the velocity field but the particle field) is described by this step-wise approach in App.A.

3 Space Orbit (Path) and Total Energy

The space coordinate x always appears in the velocity field $u_n(x)$. We can introduce the n-th step space coordinate x_n as

$$\begin{aligned} u_n(x_n) = \frac{x_{n+1} - x_n}{\tau_0} \quad &, \quad n = 0, 1, 2, \cdots \\ x_{n+1} = x_n + \tau_0 u_n(x_n) \quad &, \end{aligned} \quad (18)$$

[6] If we consider the mass m here appears in some dynamical way (, for example, through the spontaneous breakdown), $m \to \mathrm{e}^{-\varepsilon}m$ makes eq.(15) global Weyl invariant.

where x_0 is a given initial position and $\tau_0 = h\sqrt{\lambda\sigma}/m$ in eq.(7). We can trace the position of the matter-point, which was at x_0 at the initial (0-th) step, along the step process: $x_0, x_1, x_2, \cdots$. After N steps, the matter-point reaches

$$x_N = x_{N-1} + \tau_0 u_{N-1}(x_{N-1}) = x_0 + \tau_0 \sum_{n=0}^{N-1} u_n(x_n) \quad . \tag{19}$$

In terms of the continuous time,

$$x(T) = x_0 + \frac{\tau_0}{h}\int_0^T u(x(t),t)dt \quad , \tag{20}$$

where $T = N\ h, t_n = n\ h, x(t_n) = x_n, u(x,t_n) = u_n(x)$. $x(t)$ is the orbit or path of the matter-point, at x_0 initially, "moving" in the N steps.

At the n-th step, the total energy of the system, E_n, is given by

$$E_n \equiv \tilde{I}_n[u_n] \quad , \quad \text{where} \qquad \left.\frac{\delta \tilde{I}_n[u]}{\delta u(x)}\right|_{u=u_n} = 0 \quad ,$$

$$\tilde{I}_n[u] = \int dx\{\frac{\sigma}{2\tilde{\rho}_0}(\frac{du}{dx})^2 + V(u) + u\frac{dV^1}{dx} + \frac{1}{2h}(u(x+hu_{n-1}) - u_{n-1})^2\} + \tilde{I}_n^0 \ , \tag{21}$$

The system total energy E_n generally *changes* as the step number increases.

$$W(t_n) \equiv \frac{h}{\tau_0}\frac{dE(t_n)}{dt_n} \equiv \frac{\tilde{I}_{n+1}[u_{n+1}] - \tilde{I}_n[u_n]}{\tau_0} =$$

$$\frac{1}{\tau_0}\int dx \left\{\frac{\sigma}{2\tilde{\rho}_0}\left((\frac{du_{n+1}}{dx})^2 - (\frac{du_n}{dx})^2\right) + V(u_{n+1}) - V(u_n) + (u_{n+1} - u_n)\frac{dV^1}{dx}\right.$$

$$\left. + \frac{1}{2h}\{(u_{n+1} + hu_n\frac{du_{n+1}}{dx} - u_n)^2 - (u_n + hu_{n-1}\frac{du_n}{dx} - u_{n-1})^2\}\right\} + \frac{\tilde{I}_{n+1}^0 - \tilde{I}_n^0}{\tau_0}, \tag{22}$$

where $E(t_n) \equiv E_n$ and $W(t_n)$ is the energy rate. From the above formula we get the expression for the energy at $t = Nh = t_n$.

$$E(t_N) = \tilde{I}_N[u_N] = \tilde{I}_{N-1}[u_{N-1}] + \tau_0 W(t_{N-1}) = \tilde{I}_1[u_1] + \tau_0 \sum_{n=1}^{N-1} W(t_n) \quad . \tag{23}$$

When we regard the process of the increasing step-number as the time development, the system generally does *not* conserve energy. [7] $E(t_n)$ generally changes step by step. We can physically understand that the increase or

[7] We will numerically confirm the non-conservation later in Sec.6.

decrease of the total system energy is given or subtracted by the outside (environment). The energy functional (10) describes the *open-system* dynamics. When $E(t_n)$ satisfies

$$W(t_n) = \frac{h}{\tau_0}\frac{dE(t_n)}{dt_n} \quad \rightarrow \quad W_0(\text{constant}) \quad \text{as } n \rightarrow \infty \quad , \tag{24}$$

we say the system finally reaches the *steady* energy-state. [8] For the special case of $W_0 = 0$, we say the system finally reaches the *constant* energy-state [9] .

In Sec.6, we treat the $W_0 = 0$ case. As the example of the more general case ($W_0 \neq 0$), another model is given in App.C.

4 Statistical Fluctuation Effect

We are considering the system of *large number of matter-particles*, hence the physical quantities, such as energy and entropy, are given by some *statistical average*. In the present approach, the system behavior $u_n(x) = u(x, t_n)$ is completely determined by eq.(14) when the initial configuration $u_0(x) = u(x, 0)$ is given. We have obtained the solution by the *continuous* variation $\delta u(x)$ to $\tilde{I}_n[u]$, (10). In this sense, $u_n(x)$ is the 'classical path'. Here we should note that the present formalism is an *effective* way to calculate the physical properties of this *statistical* system. *Approximation* is made in the following points:

1) So far as $h \neq 0$, the *finite time-increment* gives *uncertainty* to the minimal solution $u_n(x)$. This is because we cannot specify the minimum configuration definitely, but can only do it with *finite* uncertainty.

2) The real fluid matter is made of many micro particles with small but *non-zero size*. The existence of the characteristic particle size gives uncertainty to the minimal solution in this continuum formalism. Furthermore the particle size is not constant but does distribute in the statistical way. The shape of each particle differs. The present continuum formalism has limitation to describe the real situation accurately.

3) The system energy generally changes step by step. The present model (2) describes an open-system. It means the present system *energetically* interacts with the outside. Such interaction is caused by the dissipative term in (2).

[8] Energy *constantly* comes in or goes out. App.C shows such an example.
[9] Energy does not go out and does not come in.

We claim the fluctuation comes *not* from the *quantum effect* but from the *statistics* due to the uncertainty which comes from the *finite* time-separation and the spacial distribution of *size* and *shape*.

To take into account this fluctuation effect, we *newly* define the n-th energy functional $\Gamma[u(x); u_{n-1}(x)]$ in terms of the original one $\tilde{I}_n[u(x)]$, (10), using the path-integral.

$$\mathrm{e}^{-\frac{1}{\alpha}\Gamma[u(x);u_{n-1}(x)]} = \int \mathcal{D}u(x)\mathrm{e}^{-\frac{1}{\alpha}\tilde{I}_n[u(x)]} \quad ,$$

$$\tilde{I}_n[u(x)] = \int dx\{\frac{\sigma}{2\tilde{\rho}_0}(\frac{du}{dx})^2 + V(u) + u(x)\frac{dV^1(x)}{dx} + \frac{1}{2h}(u(x+hu_{n-1}) - u_{n-1})^2\} + \tilde{I}_n^0 \quad ,$$

$$\sigma \equiv 1 \quad , \quad \tilde{\rho}_0 \equiv 1 \quad , \quad V(u) = \frac{m^2}{2}u^2 + \frac{\lambda}{4!}u^4 \quad ,(25)$$

In the above path-integral expression, *all* paths $\{u(x); -l \leq x \leq l\}$ are taken into account.

We are considering the minimal path $u_n(x)$ as the dominant configuration and the small deviation $q(x)$ around it.

$$u(x) = u_n(x) + \sqrt{\alpha}q(x) \quad , \quad |\sqrt{\alpha}q| \ll |u_n| \quad , \quad \left.\frac{\delta\tilde{I}_n[u]}{\delta u}\right|_{u=u_n} = 0 \quad , \quad (26)$$

In eq.(25) and eq.(26), a *new* expansion parameter α is introduced. ($[\alpha]=[I_n]=\mathrm{ML}^2\mathrm{T}^{-2}$) As the above formula shows, α should be small. The concrete form should be chosen depending on problem by problem. It should *not* include Planck constant, $\hbar$, because the fluctuation does not come from the quantum effect. It should be chosen as

1) the dimension is consistent,

2) it is proportional to the small scale parameter which characterizes the relevant physical phenomena such as the mean-free path of the fluid particle,

3) the precise value should be best-fitted with the experimental data.

The background-field method[5, 6] tells us to do the Taylor-expansion around u_n. [10]

$$\tilde{I}_n[u(x)] = \tilde{I}_n[u_n(x) + \sqrt{\alpha}q(x)] = \sum_{l=0}^{\infty}\alpha^{l/2}\frac{q(x)^l}{l!}\left.\frac{\delta^l\tilde{I}_n[u]}{\delta u(x)^l}\right|_{u_n} = \sum_{l=0}^{\infty}S_l[u_n] \quad , (27)$$

[10] The background-field method was originally introduced to quantize the gravitational field theory. Here we borrow the method only to define the *statistical* distribution measure. The present 'splitting' is $u = u_n + \sqrt{\alpha}q$ not $u = u_n + \sqrt{\hbar}q$.

$$e^{-\frac{1}{\alpha}\Gamma[u_n(x);u_{n-1}(x)]} = \int \mathcal{D}q(x) \exp(-\frac{1}{\alpha}(S_0 + S_2 + O(q^3))) = e^{-\frac{1}{\alpha}\tilde{I}_n[u_n(x)]} \times$$

$$\int \mathcal{D}q \exp\left[\int dx \left\{-\frac{q(x)^2}{2!}(\frac{\delta}{\delta u(x)})^2 \tilde{I}_n[u]|_{u_n} + O(q^3)\right\}\right] \quad , \quad \frac{\delta}{\delta u(x)}\tilde{I}_n[u]|_{u_n} = 0 \quad ,$$

$$S_0 = \tilde{I}_n[u_n] \quad , \quad S_1 = \int dx q(x) \left.\frac{\delta \tilde{I}_n[u]}{\delta u}\right|_{u_n} = 0 \quad ,$$

$$\frac{1}{\alpha}S_2 = \frac{q(x)^2}{2!}(\frac{\delta}{\delta u(x)})^2 \tilde{I}_n[u]|_{u_n} = \frac{1}{2}(\frac{dq}{dx})^2 + (\frac{m^2}{2} + \frac{\lambda}{2}{u_n}^2)q^2 + \frac{1}{2h}(q + hu_{n-1}\frac{dq}{dx})^2$$

$$\equiv \frac{1}{2}\frac{d}{dx}(q\frac{dq}{dx}) + \frac{1}{2}qDq + O(h) \quad , \quad D \equiv -\frac{\sigma(=1)}{\tilde{\rho}_0(=1)}\frac{d^2}{dx^2} + \lambda {u_n}^2 + m^2 + \frac{1}{h} - \frac{du_{n-1}}{dx} \quad , \quad (28)$$

where we make the Gaussian(quadratic, 1-loop) approximation. [11]

$$e^{-\frac{1}{\alpha}\Gamma[u_n(x);u_{n-1}(x)]} = e^{-\frac{1}{\alpha}\tilde{I}_n[u_n(x)]} \times (\det D)^{-1/2} \quad ,$$

$$(\det D)^{-1/2} = \exp\left\{-\frac{1}{2}\mathrm{Tr}\ln D\right\} = \exp\left\{\frac{1}{2}\mathrm{Tr}\int_0^\infty \frac{e^{-\tau D}}{\tau}d\tau + \mathrm{const}\right\} \quad , \quad (29)$$

where τ is called Schwinger's proper time[7]. ($[\tau]=[D^{-1}]=$L/M.)

To rigorously define the inside of the above exponent, we introduce Dirac's abstract state vector $|x>$ and $<x|$.

$$< x|e^{-\tau D}|y > \equiv G(x,y;\tau) \quad ,$$

$$(\frac{\partial}{\partial \tau} + D)G(x,y;\tau) = 0 \quad , \quad \lim_{\tau \to +0} G(x,y;\tau) = \delta(x-y) \quad ,$$

$$D = -\frac{\sigma(=1)}{\tilde{\rho}_0(=1)}\frac{d^2}{dx^2} - \bar{V}(x) \quad , \quad \bar{V}(x) = -\lambda u_n(x)^2 - m^2 - \frac{1}{h} - \frac{du_{n-1}}{dx} \quad , \quad (30)$$

$G(x,y;\tau)$ is called *heat-kernel*.

In App.B, we evaluate $\ln(\det D)^{-1/2} = \frac{1}{2}\int_0^\infty d\tau \mathrm{Tr}G(x,y)/\tau =$
$\frac{1}{2}\int_0^\infty d\tau \int_{-l}^{l} dx\ G(x,x)/\tau$. Up to the first order of $\bar{V}$, the result is given by

$$\frac{l}{\sqrt{\pi}}\sqrt{\epsilon\Lambda} - \frac{1}{2\sqrt{\pi\epsilon\mu}}\int_{-l}^{l} dz\epsilon(\lambda u_n(z)^2 + m^2 + \frac{1}{h} + \frac{du_{n-1}(z)}{dz}) \quad , \quad (31)$$

where the *infrared cut-off* parameter $\mu \equiv \sqrt{\sigma}/l$ and the *ultraviolet cut-off* parameter $\Lambda \equiv h^{-1}$ are introduced. [12] $\epsilon^{-1} \equiv \sigma/\tilde{\rho}_0 = 1$. We see the mass

[11] $O(h) = (h/2)u_{n-1}^2(dq/dx)^2$ may be ignored for $h \ll 1$.

[12] The last term in the round-brackets does not contribute due to the periodicity of $u_{n-1}(z)$.

parameter m^2 shifts under the influence of the fluctuation. [13]

$$m^2 \quad \to \quad m^2 + \frac{\alpha}{\sqrt{\pi\epsilon\mu}}\epsilon\lambda = m^2 + \alpha\lambda\sqrt{\frac{l\tilde{\rho}_0}{\pi\sigma\sqrt{\sigma}}} \quad , \tag{32}$$

And the bottom of the potential shifts as

$$\begin{aligned} V(u_{min}) \quad &\to \quad V(u_{min}) + \alpha\left\{\frac{1}{2}\sqrt{\frac{\epsilon\Lambda}{\pi}} - \frac{\epsilon}{2\sqrt{\pi\epsilon\mu}}(m^2 + \frac{1}{h})\right\} \\ &= V(u_{min}) + \alpha\sqrt{\frac{\tilde{\rho}_0}{4\pi\sigma}}\left\{\frac{1}{\sqrt{h}} - \sqrt{\frac{l}{\sqrt{\sigma}}}(m^2 + \frac{1}{h})\right\} \quad . \end{aligned} \tag{33}$$

The coupling λ is also shifted by the $\mathrm{O}(\bar{V}^2)$ correction. [14] The shift of these parameters corresponds to the *renormalization* in the field theory[9]. In this effective approach, we have *physical* cut-offs μ and Λ which are expressed by the (finite) parameters appearing in the starting energy-functional. When the functional (10) (effectively) works well, all effects of the statistical fluctuation reduces to the simple shift of the original parameters. This corresponds to the renormalizability condition in the field theory.

5 Boltzmann's Transport Equation

We use, for simplicity, the the original names for the shifted parameters. The step-wise development equation (14) with $\delta V/\delta u = m^2 u + \frac{\lambda}{3!}u^3 + u_{n-1}\frac{du_n}{dx}$, $V_n^1 = 0$, and the correction term (12) is written as

$$\begin{aligned} \frac{1}{h}(u_n(x) - u_{n-1}(x)) = \frac{d^2u_n}{dx^2} - m^2u_n - \frac{\lambda}{3!}{u_n}^3 - u_{n-1}\frac{du_n}{dx} \qquad \text{or} \\ u_{n-1}(x) = \frac{u_n(x) - h\{\frac{d^2u_n}{dx^2} - m^2u_n - \frac{\lambda}{3!}{u_n}^3\}}{1 - h\frac{du_n}{dx}} \quad . \end{aligned} \tag{34}$$

[13] This corresponds to renormalization of "mass" m^2 in the field theory. When natural cut-offs (IR and UV) are there in the system model-parameters, the divergences coming from the space integral and the mode summation are effectively expressed by "large" but finite quantities.

[14] The coupling (λ) shift can be obtained from $O(\bar{V})$ result (33) by assuming the "renormalization" consistently works. Noting $V(u_{min}) = -6m^4/\lambda$, λ should shift as

$$\lambda \quad \to \quad \lambda + \frac{\alpha}{6}(\frac{\lambda}{m^2})^2\sqrt{\frac{\tilde{\rho}_0}{4\pi\sigma}}\frac{1}{\sqrt{h}}\left\{1 - \sqrt{\frac{l}{\sqrt{\sigma}h}}\right\}$$

The latter form is convenient for the 'backward' recursive computation: $u_n \to u_{n-1}$. When the system reaches the *equilibrium state* after sufficient recursive computation ($n \gg 1$), we may assume $u_{n-1}(x) = u_n(x) \equiv u^\infty(x)$. $u^\infty(x)$ satisfies

$$\frac{d^2u^\infty}{dx^2} - m^2u^\infty - \frac{\lambda}{3!}u^{\infty 3} - u^\infty\frac{du^\infty}{dx} = 0 \quad , \tag{35}$$

We here introduce the *distribution function* $f_n(x,v)$ as the probability for the matter-point particle in the space interval $x \sim x+dx$ and the velocity interval $v \sim v+dv$, at the step n, is given by

$$\frac{1}{\bar{N}_n} f_n(x,v)dxdv \quad , \tag{36}$$

where $\bar{N}_n$ is the total particle number of the system at the step n.[8] Then the n-th *distribution* $f_n(x,v)$ and the *equilibrium distribution* $f^\infty(x,v)$ are introduced as

$$\begin{aligned} u^\infty(x) = \frac{1}{\rho_\infty(x)}\int vf^\infty(x,v)dv \quad , \quad u_n(x) = \frac{1}{\rho_n(x)}\int vf_n(x,v)dv \quad , \\ u_n(x) \to u^\infty(x) \quad \text{and} \quad f_n(x,v) \to f^\infty(x,v) \quad \text{as} \quad n \to \infty \quad , \end{aligned} \tag{37}$$

where $u^\infty(x)$ is the *equilibrium* velocity distribution. $\rho_n(x)$ is the *particle number* density. [15] The continuity equation is given by

$$\frac{1}{h}(\rho_n(x) - \rho_{n-1}(x)) + \frac{d}{dx}(\rho_n(x)u_n(x)) = 0 \quad . \tag{38}$$

The recursion relation (34) is expressed, in terms of the distribution functions, as

$$\begin{aligned} \frac{1}{h}\left[f_n(x+hu_{n-1}(x),v) - f_{n-1}(x,v)\right] = \\ \frac{\partial^2 f_n(x,v)}{\partial x^2} - m^2 f_n(x,v) - \frac{\lambda}{3!}f_n(x,v)u_n(x)^2 \quad , \\ \text{where} \quad u_n(x) = \frac{1}{\rho_n(x)}\int vf_n(x,v)dv \quad , \end{aligned} \tag{39}$$

This is the *Boltzmann's transport equation* for the 2-body and 4-body velocity-interactions. We can express the step-wise expression (39) in the continuous

[15] Here we list the physical dimension of some quantities appearing this section. $[x]$=L, $[u_n]$=$[v]$=L/T, $[k_B\mathcal{T}]$=ML^2/T^2, $[\rho_n(x)]$=L^{-1}, $[\tilde{\rho}_n(x)]$=ML^{-1}, $[f_n]$=T/L^2, $[P_n]$=ML/T^2, $[q_n]$=ML^2/T^3.

time t form as in Sec.2. This is the integrodifferential equation for $f_n(x,v)$ when ρ_n is a constant. The right hand side (RHS) is called *collision term*. We notice when we may replace u_{n-1}, in the LHS of eq.(39), by u_n, the above recursion relation determine the (n-1)th step distribution f_{n-1} by the n-th step data, f_n and u_n. [16]

We now introduce some physical quantities used in the *non-equilibrium statistical mechanics*. The *entropy* S_n and the *total particle-number* $\bar{N}_n$ are defined by

$$S_n \equiv -k_B \int dv \int dx f_n(x,v) \ln f_n(x,v) \quad ,$$
$$\bar{N}_n = \int dx \rho_n(x) = \int dx \int dv f_n(x,v) \quad , \quad \rho_n(x) = \int dv f_n(x,v) \quad , \qquad (40)$$

where k_B is Boltzmann's constant. The momentum conservation at each point, x, requires

$$0 = \tilde{\rho}_n(x) \int dv (v - u_n(x)) f_n(x,v) \quad , \quad u_n(x) = \frac{1}{\rho_n(x)} \int dv \; v \; f_n(x,v) \quad . (41)$$

Besides the particle-number density $\rho_n(x)$, we have introduced the *mass* density $\tilde{\rho}_n(x)$ at step n. [17]

We here consider the case of *one kind* particle.

$$\frac{\tilde{\rho}_n(x)}{\rho_n(x)} = m_1 \text{ (constant)} \quad , \quad m_1 = m^2 \times (2l) \quad , \qquad (42)$$

where m_1 is the particle mass. [18] In this case the total mass M_n is given by

$$M_n \equiv \int dx \tilde{\rho}_n(x) = m_1 \int dx \rho_n(x) = m_1 \bar{N}_n \quad . \qquad (43)$$

The *temperature* distribution $\mathcal{T}_n(x)$, the *heat current* distribution $q_n(x)$ and the *pressure* $P_n(x)$ are given by

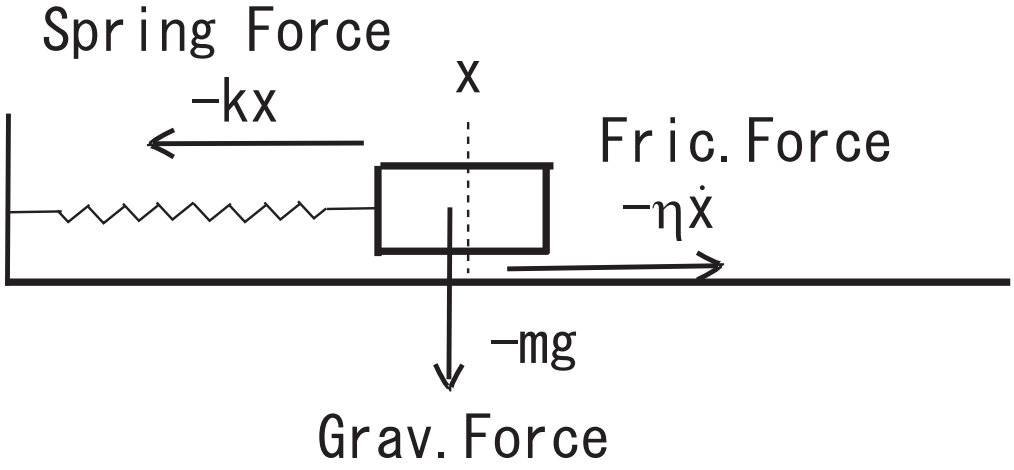
$$\frac{1}{2} k_B \mathcal{T}_n(x) \equiv \frac{1}{\rho_n(x)} \int dv \frac{m_1}{2} (v - u_n(x))^2 f_n(x,v) \quad ,$$
$$q_n(x) \equiv \int dv \frac{m_1}{2} (v - u_n(x))^3 f_n(x,v) \quad ,$$
$$P_n(x) \equiv m_1 \int dv (v - u_n(x))^2 f_n(x,v) \quad , \qquad (44)$$

[16] Solving the recursion equation (39) in the backward way, with respect to the step number n, is convenient in practice.

[17] In Sec.2, we considered the case of $\sigma = 1$ and $\tilde{\rho} = \tilde{\rho}_0 = 1$ for simplicity. When we generalize it, we have to use some equations appearing in this section to determine the form of $\sigma_n(x)$ and $\tilde{\rho}_n(x)$. Especially the viscosity $\sigma_n(x)$ depends on the temperature $\mathcal{T}_n(x)$ introduced in (44).

[18] For k kinds particles, we introduce ρ_n and $\tilde{\rho}_n$ fields for each one: $\{{\rho_n}^i, \tilde{\rho}_n^i | i = 1, 2, \cdots, k\}$ with the relation $\tilde{\rho}_n^i(x)/{\rho_n}^i(x) = m_i$.

Figure 2: The harmonic oscillator with friction, (49).



where k_B is Boltzmann's constant.

Using the transport equation (39) which $f_n(x, v)$ satisfies, we expect the following two equations are satisfied. The first one is the rephrasing the Navier-Stokes equation (34).

$$\frac{1}{h}(u_n(x) - u_{n-1}(x)) + u_{n-1}\frac{du_n}{dx} = -\frac{1}{\tilde{\rho}_n(x)}\frac{dP_n(x)}{dx} \quad . \tag{45}$$

The second one is the energy equation.

$$\frac{1}{2}\rho_n(x)k_B\{\frac{1}{h}(\mathcal{T}_n(x) - \mathcal{T}_{n-1}(x)) + u_{n-1}\frac{d\mathcal{T}_n}{dx}\} = -\frac{dq_n(x)}{dx} - P_n(x)\frac{du_n(x)}{dx} \quad .(46)$$

In the remaining sections, we present an alternative approach to the distribution function $f_n(x, v)$.

6 Classical and Quantum Mechanics and Its Trajectory Geometry

We can treat the classical mechanics and its quantization (the quantum mechanics, not the quantum field theory) in the same way. In this case, the model is simpler than the previous case (space-field theory) and we can see the *geometrical structure* clearly. Let us begin with the energy function of a system variable , x, (1 degree of freedom). For example the *position* (in

1 dimensional space) of the harmonic oscillator with friction. We take the following n-th energy function to define the step flow.

$$K_n(x) = V(x) + \frac{\eta}{2h}(x - x_{n-1})^2 + \frac{m}{2h^2}(x - 2x_{n-1} + x_{n-2})^2 + K_n^0 \quad , \quad (47)$$

where $V(x)$ is the general potential and K_n^0 is a constant which does not depend on x. For the harmonic oscillator $V(x) = kx^2/2$ where k is the spring constant. η is the friction coefficient and m is the particle mass. [19] We assume x_{n-1} and x_{n-2} are given values. As in Sec.2, the n-th step position x_n is given by the *minimal principle* of the n-th energy function $K_n(x)$: $\delta K_n = 0, x \to x + \delta x$.

$$\left.\frac{\delta V}{\delta x}\right|_{x=x_n} + \frac{\eta}{h}(x_n - x_{n-1}) + \frac{m}{h^2}(x_n - 2x_{n-1} + x_{n-2}) = 0 \quad , \quad (48)$$

With the time t_n (7), the continuous limit ($h \to 0$) gives us

$$\frac{dV(x)}{dx} + \eta\frac{dx}{dt} + m\frac{d^2x}{dt^2} = 0 \quad , \quad (49)$$

where $t_n = nh \to t$, $x_n = x(t_n) \to x(t)$, $(x_n - x_{n-1})/h = dx/dt|_{t_n} \to dx/dt$, $(x_n - 2x_{n-1} + x_{n-2})/h = d^2x/dt^2|_{t_n} \to d^2x/dt^2$. For the case of $V = kx^2/2$, this is the harmonic oscillator with the friction η. See Fig.2. This is a simple *dissipative* system. [20]

The recursion relation (48) gives us, for the initial data x_0 and x_1, the series $\{ x_n = x(t_n) | n = 0, 1, 2, \cdots \}$. This is the classical 'path'. The fluctuation of the path comes from the *uncertainty principle* of the *quantum mechanics* in this case. (We are treating the system of 1 degree of freedom. No statistical procedure is necessary.) As the time-interval h tends to zero, the energy uncertainty grows ($\Delta t \cdot \Delta E \geq \hbar$). Hence the path x_n, obtained by the recursion relation (48), has more uncertainty as h goes to 0. As in Sec.4, we can generalize the n-th energy function $K_n(x)$, (47), to the following one

[19] Here we list the dimension of parameters and variables in this section. $[x]$=$[x_n]$=L, $[v]$=$[v_n]$=L/T, $[t]$=$[t_n]$=T, $[q]$=$\mathrm{T}^{1/2}\mathrm{M}^{-1/2}$, $[\hbar]$=ML^2/T, $[m]$=M, $[\eta]$=M/T, $[h]$=T, $[K_n]$=$[V]$=$\mathrm{ML}^2/\mathrm{T}^2$, $[k]$=M/T^2, $[\sqrt{\eta h}]$=$\mathrm{M}^{1/2}$, $[\sqrt{mh^2}]$=$\mathrm{M}^{1/2}\mathrm{T}$. Some ones, such as t, m, h, q and V, appearing before this section have different dimensions in this section.

[20] The eq. (49) is compared with the eq. (9) for the case of the no external force:

$$\tilde{\rho}_0\frac{\delta V(u(x,t))}{\delta u(x,t)} - \sigma\frac{\partial^2 u(x,t)}{\partial x^2} + \tilde{\rho}_0\frac{\partial u(x,t)}{\partial t} = 0 \quad ,$$

where we notice the friction term in eq.(49) corresponds to the dissipative term in eq.(9).

$\Gamma(x_{n-1}, x_{n-2})$ in order to take into account the quantum effect.

$$e^{-\frac{1}{\hbar}h\Gamma(x_{n-1},x_{n-2})} = \int_{-\infty}^{\infty} dx\ e^{-\frac{1}{\hbar}hK_n(x)} \quad ,$$

$$K_n(x) = V(x) + \frac{\eta}{2h}(x - x_{n-1})^2 + \frac{m}{2h^2}(x - 2x_{n-1} + x_{n-2})^2 + K_n^0 \quad . \quad (50)$$

We can evaluate the quantum effect by the expansion around the classical value $x_n : x = x_n + \sqrt{\hbar}\ q$ where $\hbar$ is Planck constant. [21]

$$e^{-\frac{1}{\hbar}h\Gamma(x_n;x_{n-1},x_{n-2})} = \int dx\ e^{-\frac{1}{\hbar}hK_n(x)} = \int dq\ e^{-\frac{1}{\hbar}hK_n(x_n+\hbar q)} \quad ,$$

$$\left.\frac{\delta K_n}{\delta x}\right|_{x=x_n} = \left.\frac{\delta V}{\delta x}\right|_{x=x_n} + \frac{\eta}{h}(x_n - x_{n-1}) + \frac{m}{h^2}(x_n - 2x_{n-1} + x_{n-2}) = 0 \quad ,$$

$$\Gamma_n \equiv \Gamma(x_n; x_{n-1}, x_{n-2}) = K_n(x_n) + \frac{\hbar}{2h}\ln(k + \frac{\eta}{h} + \frac{m}{h^2}) \quad , \quad (51)$$

where the final expression is for the oscillator model: $V = kx^2/2$. The quantum effect does not depend on the step number n. It means the quantum effect contributes to the energy as an additional fixed constant at each step.

The energy rate is obtained as

$$h\frac{dK(t_n)}{dt_n} \equiv K_{n+1}(x_{n+1}) - K_n(x_n) = \Gamma_{n+1} - \Gamma_n \equiv h\frac{d\Gamma(t_n)}{dt_n}$$
$$= V(x_{n+1}) - V(x_n) + \frac{\eta}{2h}\{(x_{n+1} - x_n)^2 - (x_n - x_{n-1})^2\}$$
$$+\frac{m}{2h^2}\{(x_{n+1} - 2x_n + x_{n-1})^2 - (x_n - 2x_{n-1} + x_{n-2})^2\}K_{n+1}^0 - K_n^0 \quad . \quad (52)$$

The present system is again an open system, and the energy generally changes.

In terms of the position difference $x_n - x_{n-1} \equiv \Delta x_n$ and the velocity difference $(x_n - 2x_{n-1} + x_{n-2})/h \equiv v_n - v_{n-1} \equiv \Delta v_n$, we can rewrite the energy at n-step and read the *metric* as follows. [22]

$$K_n(x_n) = V(x_n) + \frac{\eta}{2h}(x_n - x_{n-1})^2 + \frac{m}{2h^2}(x_n - 2x_{n-1} + x_{n-2})^2 + K_n^0$$
$$= \frac{1}{h^2}\{V(x_n)(\Delta t)^2 + \frac{\eta h}{2}(\Delta x_n)^2 + \frac{mh^2}{2}(\Delta v_n)^2\} + K_n^0 \quad , \quad (53)$$

where h (time increment) in the first term within the round brackets is replaced by Δt. This shows us the metric for the n-step energy function is given by

$$(\Delta s_n)^2 \equiv 2h^2 K_n(x_n) = 2V({x_n}'/\sqrt{\eta h})(\Delta t)^2 + (\Delta {x_n}')^2 + (\Delta v_n')^2 \quad ,$$
$${x_n}' \equiv \sqrt{\eta h}x_n \quad , \quad {v_n}' \equiv \sqrt{mh^2}v_n \quad , \quad (54)$$

[21] Do not confuse $\hbar$ (Planck constant/2π) with h (time interval).

[22] $v_n \equiv (x_n - x_{n-1})/h\ = \Delta x_n/h$

where , for the oscillator model, $V(x_n'/\sqrt{\eta h}) = (k'/2){x_n'}^2, k' \equiv k/\eta h$. Eq.(54) shows the energy-line element Δs^2 in the (t, x_n', v_n') space. [23] Note that the above metric is *along the path* $x_n = x(t_n),\ v_n = v(t_n) = (x(t_n) - x(t_{n-1}))/h$ given by (48). The metric is used, in the next section, as the *geometrical* basis for fixing the statistical ensemble.

We take the freedom of the value K_n^0 in the following way.

$$K_n^0 = -V(x_n) - \frac{m}{2h^2}(x_n - 2x_{n-1} + x_{n-2})^2 + V(x_0) + \frac{m}{2h^2}(x_1 - x_0)^2 \quad . \ (55)$$

This is chosen in such a way that the step n energy $K_n(x_n)$, for the no dissipation ($\eta = 0$), does *not* depend on the step number n and the value is the total energy at the initial step (last 2 terms in (55)). The graphs of movement and energy change, for various viscosities, are shown in Fig.3-9. For the no friction case, the oscillator keeps the initial energy (Fig.4). When the viscous effect appears, the *energy changes step by step*, and finally reaches a constant *nonzero* value (Fig.6,Fig.7, Fig.9). We understand the finally-remaining energy (constant) as the dissipative one. Physically (in the real matter) the energy is realized as the pressure and the temperature which characterize the particle's "environment"(out-side world). [24] For the resonate case ($4k/m = (\eta/m)^2$), both the movement and the energy are large.

7 Statistical Ensemble, Geometry and Initial Condition

In this section, we consider some statistical ensemble of the classical mechanical system taken in the previous section. Namely, we take N 'copies' of the classical model and regard them as a set of (1 dimensional) particles, where the dynamical configuration distributes in the probabilistic way. N is a large number. [25] The set has N degrees of freedom: $x_1, x_2, \cdots, x_N$. As the physical systems, (1 dimensional) *viscous* gas and *viscous* liquid are examples. [26] Each particle obeys the (step-wise) Newton's law (48) with different *initial conditions*. N is so large that we do not or can not observe the initial data. Usually we do not have interest in the trajectory of every particle

[23] In eq.(54), the first term shows the potential part, the second one the kinetic part and the third one a new term. In ref.[10] and ref.[11], the hysteresis term appears as a new one.

[24] Note that the particle does *not* move at the final stage.

[25] For example, $N \sim 10^{23}$.

[26] We are considering N-body problem where each particle moves (fluid flows) with friction. We aproach it using the effective 1-body energy function (47).

Figure 3: Harmonic oscillator with no friction, Movement, h=0.1,$\sqrt{k/m}$=0.01, η/m=0.0, total step no =20000. The step-wise solution (48) correctly reproduces the analytic solution: $x(t) = \sin(\omega t)$, $0 \leq t \leq 2000$, $x(0) = 0,\ \dot{x}(0) = 1$.

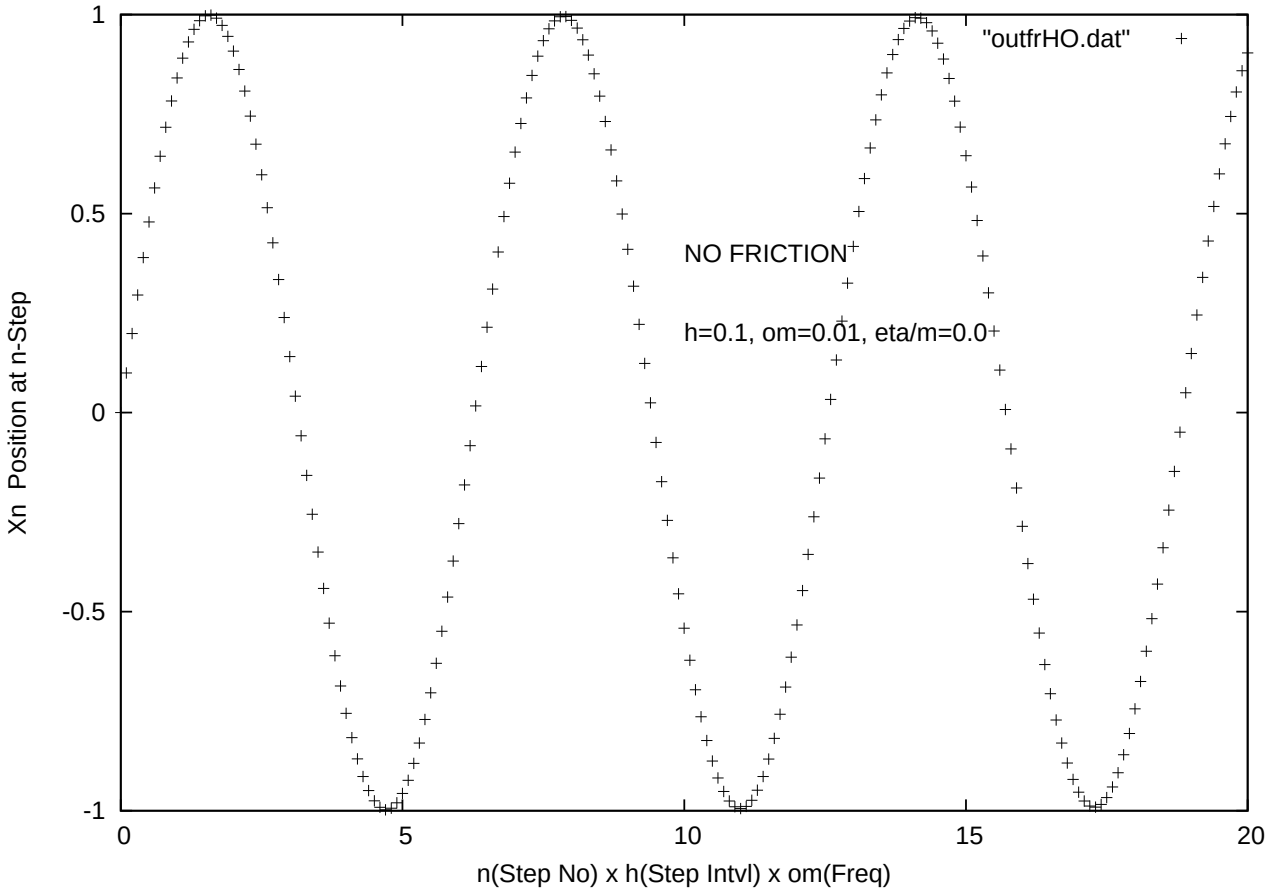


Figure 4: Harmonic oscillator with no friction, Energy change, h=0.1,$\sqrt{k/m}$=0.01, η/m=0.0, total step no =20000.

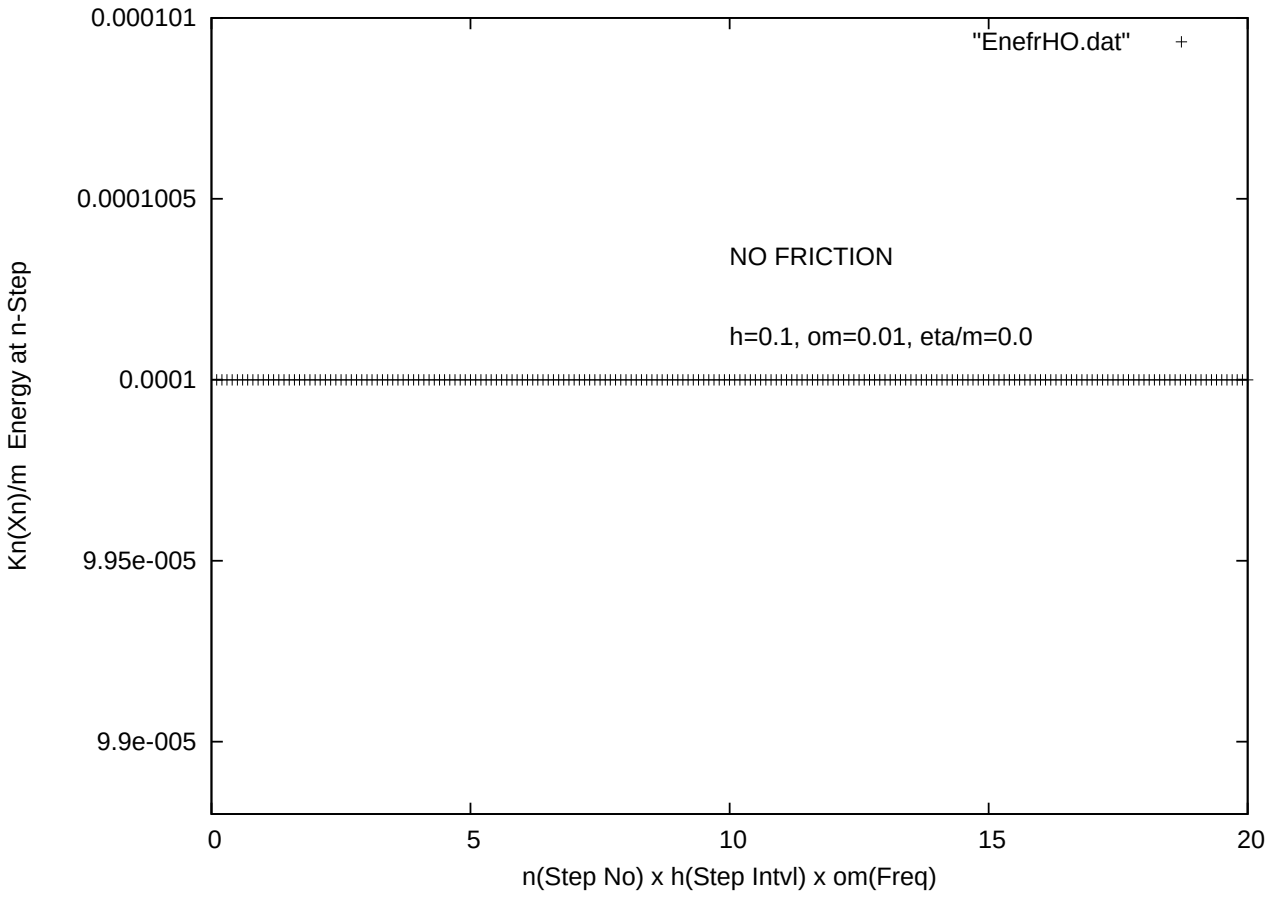


Figure 5: Harmonic oscillator with friction, Movement, h=0.1,$\sqrt{k/m}$=0.01, η/m=0.005 (Elasticity dominate) and 0.03 (Viscosity dominate), total step no =20000. The step-wise solution (48) correctly reproduces the analytic solution: (1)Elast.Dom. $x(t) = \mathrm{e}^{-\eta' t/2} \sin(\sqrt{4\omega^2 - \eta'^2}\ t/2)$, $0 \le t \le 2000$, $x(0) = 0,\ \dot{x}(0) = \sqrt{4\omega^2 - \eta'^2}/2 = (1.94 \times 10^{-2})/2$ (2)Visco.Dom. $x(t) = \mathrm{e}^{-\eta' t/2} \sinh(\sqrt{\eta'^2 - 4\omega^2}\ t/2)$, $0 \le t \le 2000$, $x(0) = 0,\ \dot{x}(0) = \sqrt{\eta'^2 - 4\omega^2}/2 = (2.24 \times 10^{-2})/2$.

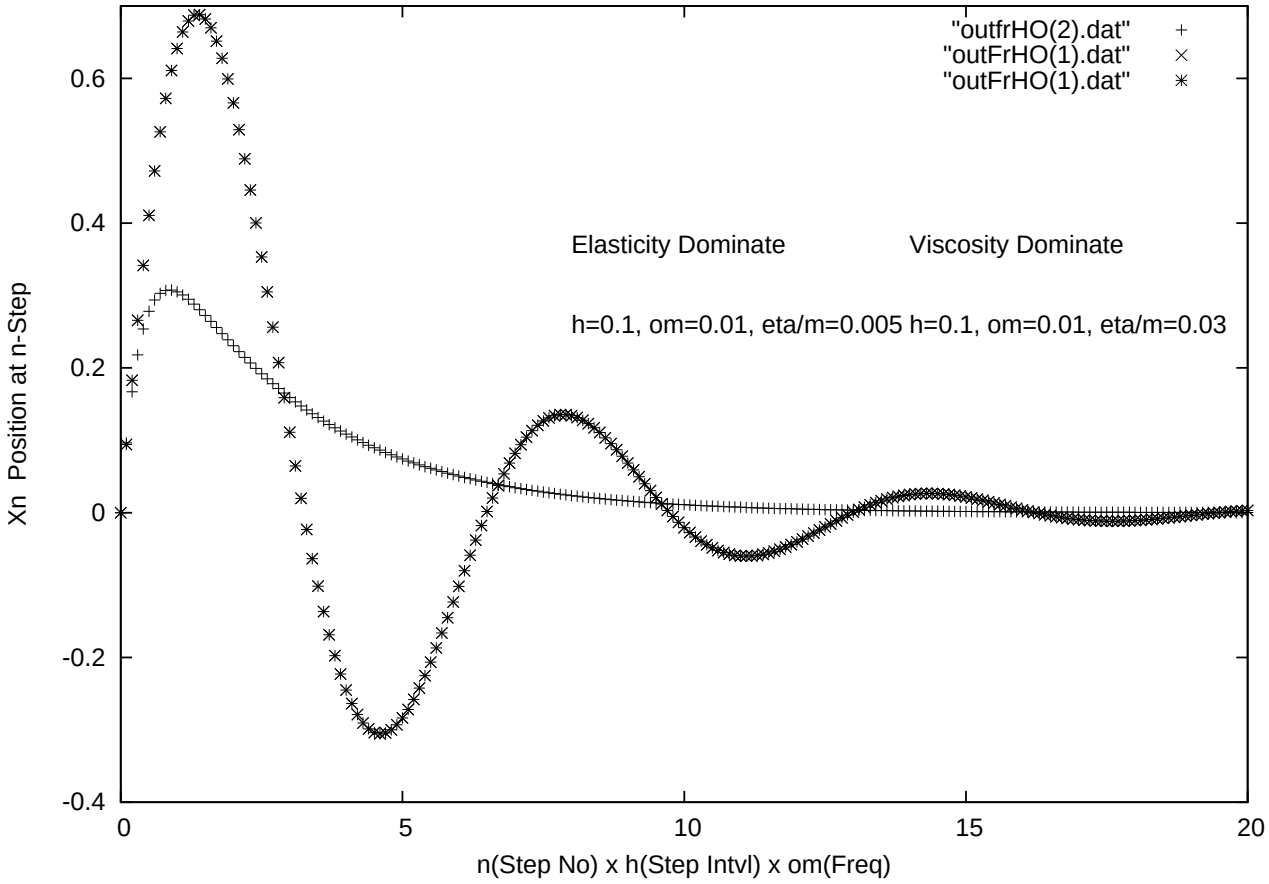


Figure 6: Harmonic oscillator with friction, Energy change, h=0.1,$\sqrt{k/m}$=0.01, η/m=0.005 (Elasticity dominate) , total step no =20000.

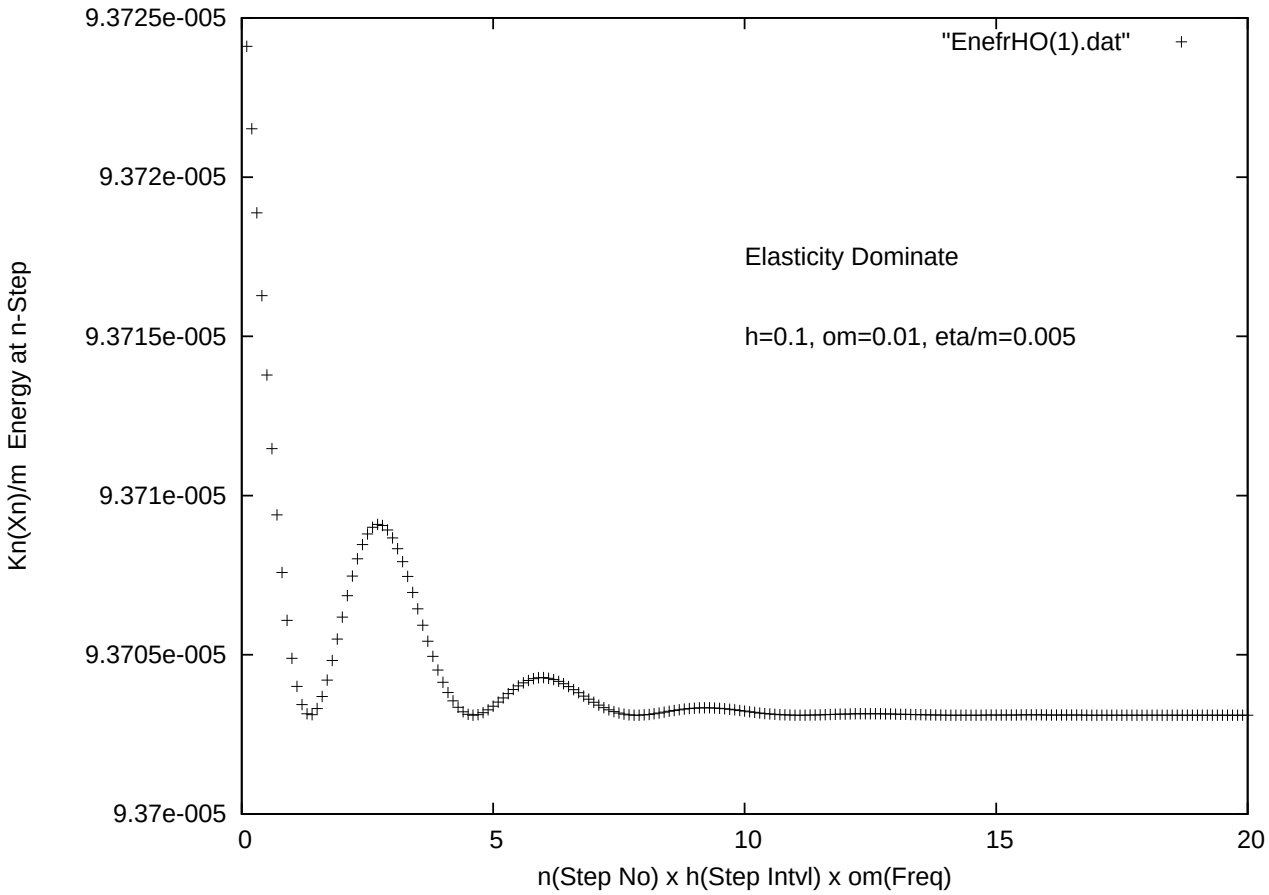


Figure 7: Harmonic oscillator with friction, Energy change, h=0.1,$\sqrt{k/m}$=0.01, η/m=0.03 (Viscosity dominate), total step no =20000.

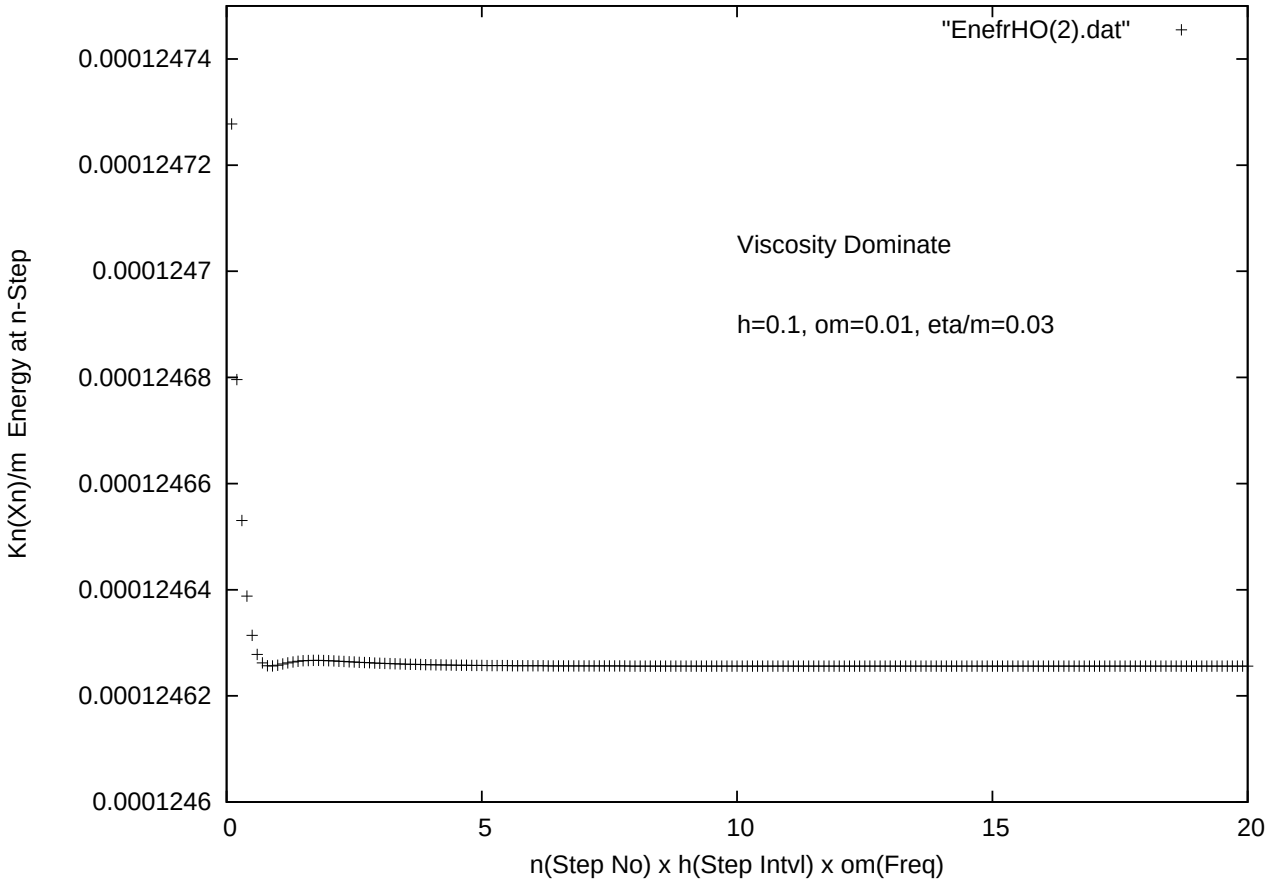


Figure 8: Harmonic oscillator with friction, Movement, h=0.1,$\sqrt{k/m}$=0.01, η/m=0.02 (Resonant), total step no =20000. The step-wise solution (48) correctly reproduces the analytic solution: $x(t) = te^{-\omega t}$, $0 \leq t \leq 2000$, $x(0) = 0,\ \dot{x}(0) = 1$.

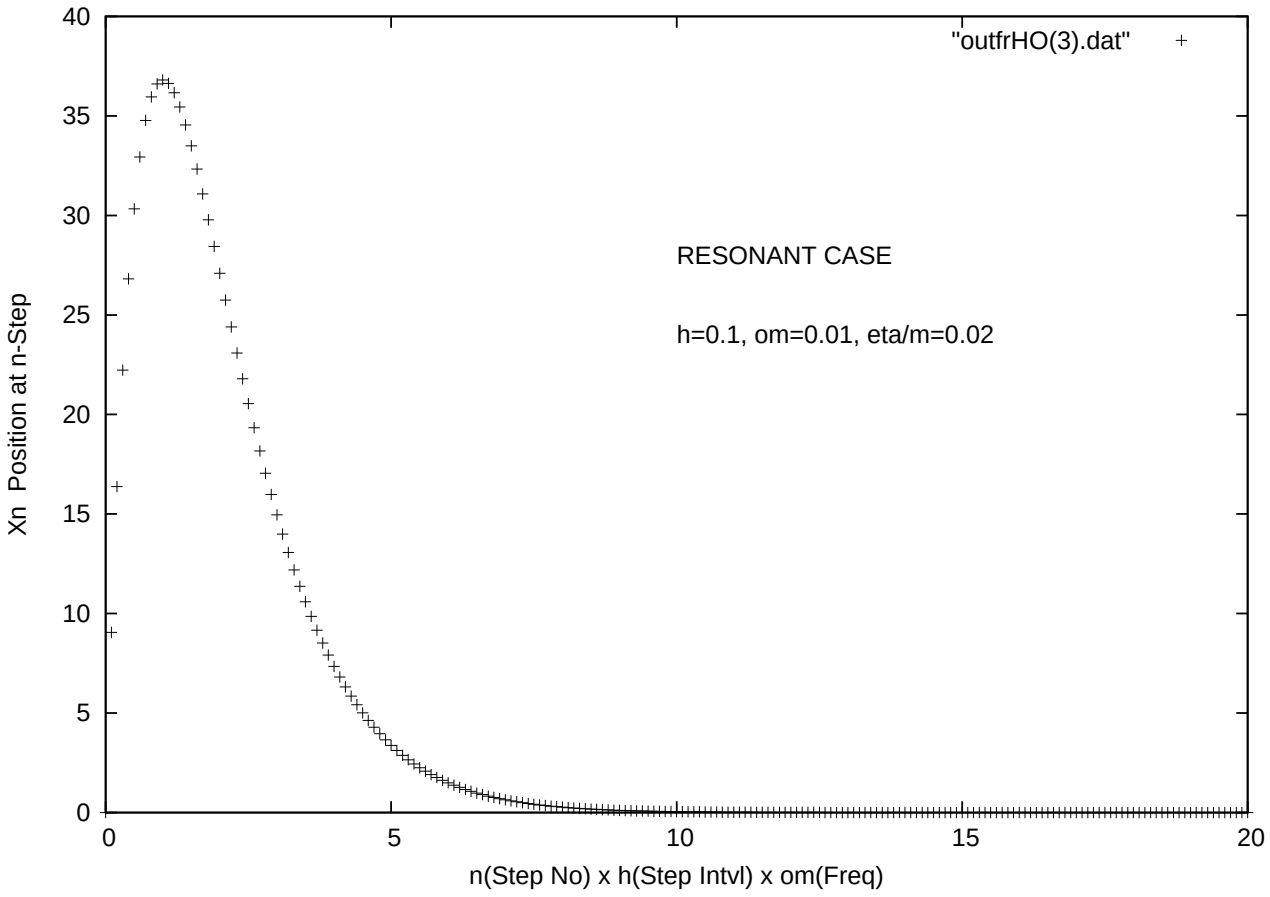


Figure 9: Harmonic oscillator with friction, Energy change, h=0.1,$\sqrt{k/m}$=0.01, η/m=0.02 (Resonant), total step no =20000.

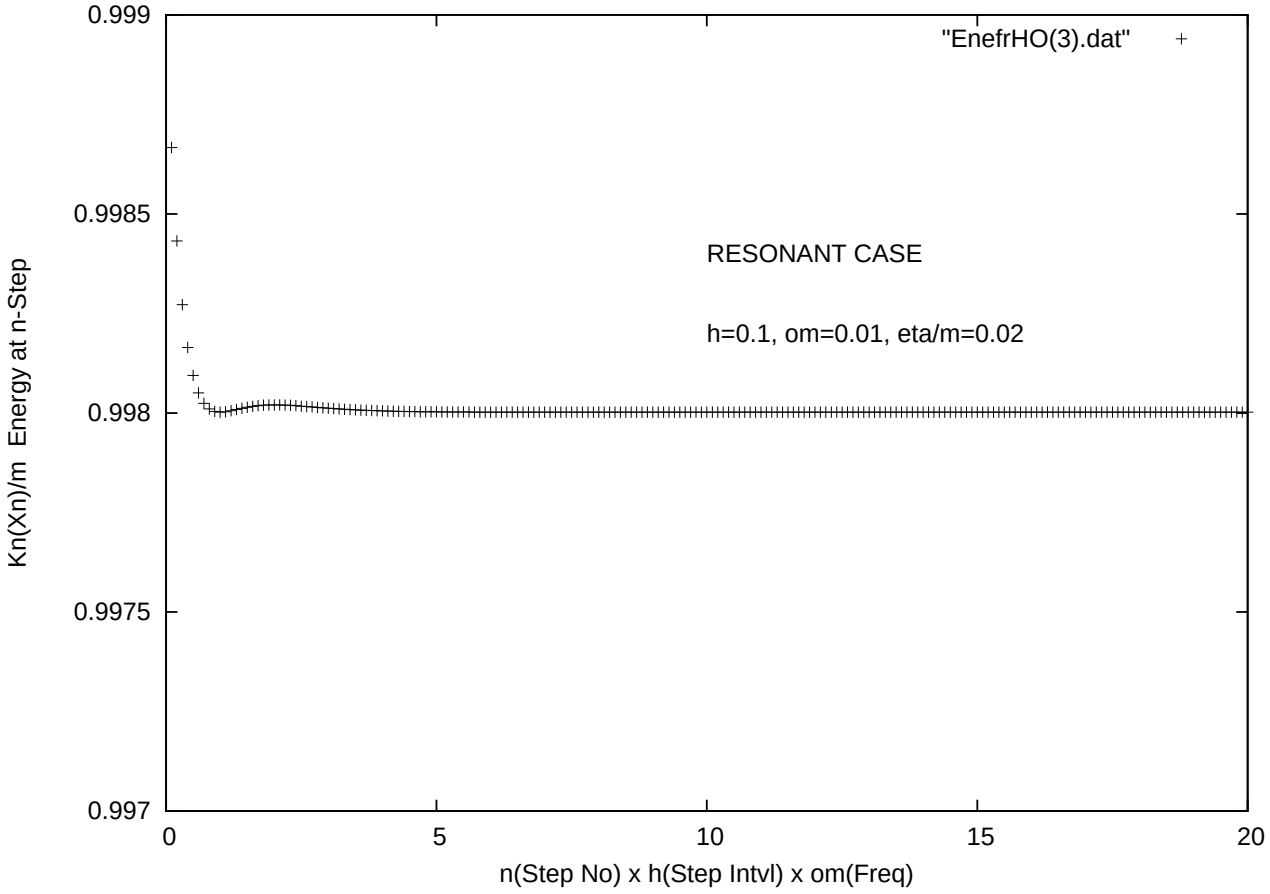
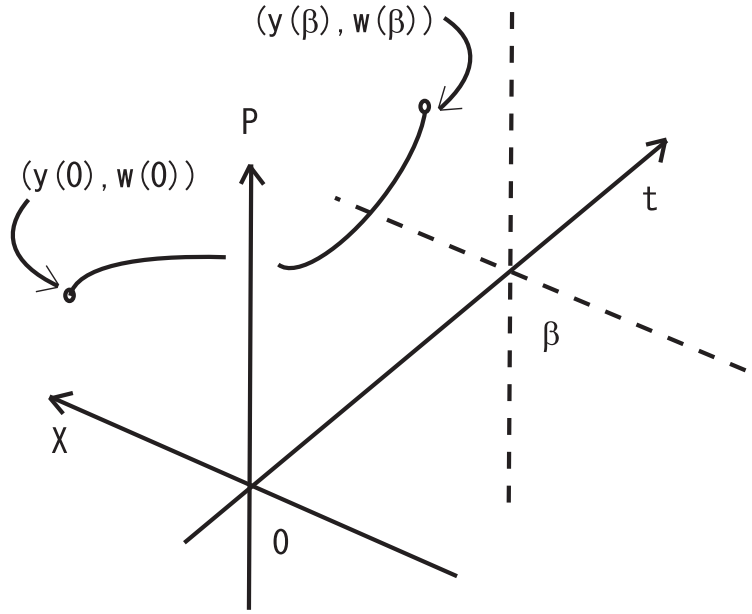


Figure 10: The path $\{(y(t), w(t), t)|0 \leq t \leq \beta\}$ of line in 3D bulk space (X,P,t).



and do not observe it. We have interest only in the macroscopic quantities such as *total energy* and *total entropy*. The N particles (fluid molecules) in the present system are weakly interacting each other in such way that each particle almost independently moves except that energy is exchanged.

As the statistical ensemble, we adopt the Feynman's idea of "path-integral" [12, 13, 14, 15, 16, 17, 11, 10]. We take into account all possible paths $\{y_n\}$. $\{y_n\}$ need not satisfy (48) nor certain initial condition. As the *measure* for the summation (integral) over all paths, we propose the following ones based on the geometry of (54). As the first measure, we construct it in terms of the *length*, using the "Dirac-type" metric [11, 10].

$$(ds^2)_D \equiv 2V(X)dt^2 + dX^2 + dP^2 \quad - \text{on-path}(X = y(t), P = w(t)) \rightarrow (2V(y) + \dot{y}^2 + \dot{w}^2)dt^2 \quad ,$$

$$L_D = \int_0^\beta ds|_{on-path} = \int_0^\beta \sqrt{2V(y) + \dot{y}^2 + \dot{w}^2}dt = h\sum_{n=0}^{\beta/h} \sqrt{2V(y_n) + \dot{y}_n^2 + \dot{w}_n^2} \quad ,$$

$$d\mu = \mathrm{e}^{-\frac{1}{\alpha}L_D}\mathcal{D}y\mathcal{D}w \quad , \quad \mathrm{e}^{-\beta F} = \int \prod_n dy_n dw_n \mathrm{e}^{-\frac{1}{\alpha}L_D} \quad , \tag{56}$$

where α is a parameter with the dimension of length ([α]=L). See Fig.10. As explained in Sec.4, it is appropriately chosen problem by problem. β is

introduced to restrict the t-region ($0 \leq t \leq \beta$) and, in this context, should be regarded as a part of the choice of the ensemble. β plays the role of the inverse temperature. [27] Among all possible paths $\{y_n\}$, the minimal length ($\delta L_D = 0$) solution, (48), gives the dominant path $\{x_n\}$.

The second choice is constructed using the "Standard-type" metric.

$$(ds^2)_S \equiv \frac{1}{dt^2}[(ds^2)_D]^2 \quad - \text{on-path}(X = y(t), P = w(t)) \rightarrow (2V(y) + \dot{y}^2 + \dot{w}^2)^2 dt^2 \quad ,$$

$$L_S = \int_0^\beta ds|_{on-path} = \int_0^\beta (2V(y) + \dot{y}^2 + \dot{w}^2)dt = h\sum_{n=0}^{\beta/h}(2V(y_n) + \dot{y}_n^2 + \dot{w}_n^2) \quad ,$$

$$d\mu = \mathrm{e}^{-\frac{1}{\alpha}L_S}\mathcal{D}y\mathcal{D}w \quad , \quad \mathrm{e}^{-\beta F} = \int \prod_n dy_n dw_n \mathrm{e}^{-\frac{1}{\alpha}L_S} = (\text{const}) \int \prod_{n=0}^{\beta/h} dy_n \mathrm{e}^{-\frac{h}{\alpha}(2V(y_n)+\dot{y}_n^2)} \quad ,(57)$$

where we should notice dt ($= h > 0$) is non-zero. In both cases above we take the metric of the 3 dimensional (bulk) space-time (X, P, t) , which is consistently chosen with the trajectory metric (54). Note that the standard case is exactly the same expression as the free energy (trace of the density matrix) expression in the Feynman's textbook[18].

Another choice of path is making use of *surfaces* instead of *lines*. Let us consider the following 2 dim surface in the 3 dim manifold (X, P, t).

$$X^2 + P^2 = r^2(t) \quad , \quad 0 \leq t \leq \beta \quad , \tag{58}$$

where $r(t)$ is arbitrary (non-negative) function of t. We respect here the isotropy of the 2 dim phase space (X, P). See Fig.11. By varying the form of $\{r(t) : \ 0 \leq t \leq \beta\}$, we obtain different surfaces. Regarding each of them as a path used in the Feynman's path-integral, we obtain the following statistical ensemble. First the *induced metric* g_{ij} on the surface (58) is given as

$$(ds^2)_D\Big|_{\text{on-path}} = 2V(X)dt^2 + dX^2 + dP^2\Big|_{\text{on-path}} = \sum_{i,j=1}^{2} g_{ij}dX^i dX^j \quad ,$$

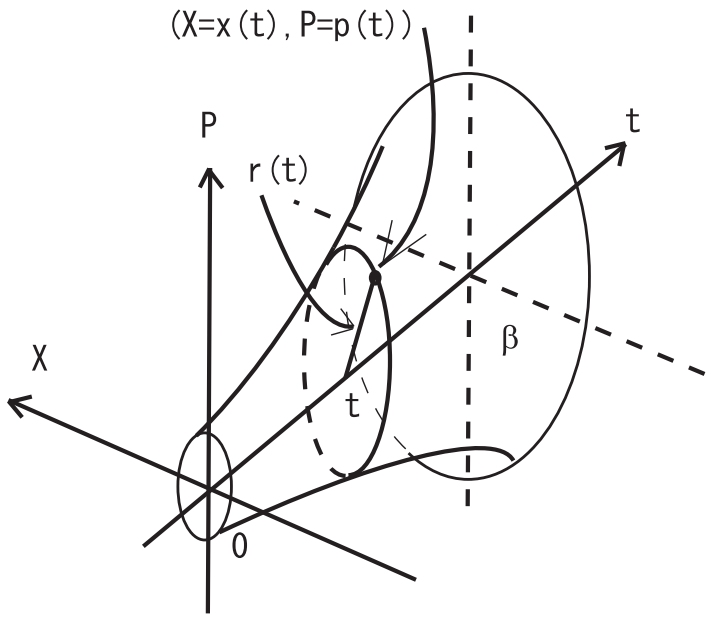
$$(g_{ij}) = \begin{pmatrix} 1 + \frac{2V}{r^2\dot{r}^2}X^2 & \frac{2V}{r^2\dot{r}^2}XP \\ \frac{2V}{r^2\dot{r}^2}PX & 1 + \frac{2V}{r^2\dot{r}^2}P^2 \end{pmatrix} \quad ,(59)$$

where $(X^1, X^2) = (X, P)$. Then the *area* of the surface (58) is given by

$$A = \int \sqrt{\det g_{ij}}d^2X = \int \sqrt{1 + \frac{2V}{\dot{r}^2}}dXdP \quad , \tag{60}$$

[27] $\beta/h = N$ should be an (large) integer. The increment h is the (inverse) temperature unit as well as the time unit. From the dimensional analysis ${k_B}^{-1}\eta\tilde{l}^2/\beta$ corresponds to the temperature. Here k_B is Boltzmann's constant and $\eta\tilde{l}^2 \equiv \hbar'$ is the combination of the friction coefficient η and some length scale $\tilde{l}$ ([$\tilde{l}$]=L) such as the mean free path of the fluid particle. Note that $\hbar'$ has the same dimension as $\hbar$. [$\hbar'$]=[$\hbar$]=ML^2/T.

Figure 11: The two dimensional surface, (58), in 3D bulk space (X,P,t).



We consider all possible surfaces of (58). The statistical distribution is, using the *area* A, given by

$$\mathrm{e}^{-\beta F} = \int_0^\infty d\rho \int_{\substack{r(0)=\rho \\ r(\beta)=\rho}} \prod_t \mathcal{D}X(t)\mathcal{D}P(t)\mathrm{e}^{-\frac{1}{\alpha}A} \quad , \tag{61}$$

In relation to Boltzmann's equation (Sec.5), we have directly defined the distribution function $f(t,x,v)$ using the geometrical quantities in the 3 dim bulk space.

8 Conclusion

We have presented the field theory approach to Boltzmann's transport equation. The collision term is explicitly obtained. Time is *not* used, instead the step number n plays the role. We have presented the n-th energy functional (10) which gives the step n configuration $u_n(x)$ from the minimal energy principle. We regard the step flow (the increase of n) as the evolution of the system, namely, *time-development*. Navier-Stokes equation is obtained by identifying time t as nh (7). Fluctuation effect due to the micro structure and micro (step-wise) movement is taken into account by generalizing

the n-th energy functional $\tilde{I}_n[u(x)]$, (10), to $\Gamma[u(x); u_{n-1}(x)]$, (25), where the classical path $u_n(x)$ is dominant but all possible paths are taken into account (path-integral). *Renormalization* is explicitly done. The total energy generally does *not* conserve. The system is an open one, namely, the energy comes in from or go out to the outside. In the latter part we have presented a direct approach to the distribution function $f_n(x, v)$ based on the geometry emerging from the mechanical (particle-orbit) dynamics. We have examined the *dissipative* system using the *minimal (variational) principle* which is the key principle in the standard field theory[19].

9 Appendix A (3+1)D Scalar Field Theory

3+1 dimensional scalar field is here treated in the present step-wise formalism. We start with the following n-th step energy functional.

$$I_n[\phi(\mathbf{x})] = \int d^3x \{\frac{1}{2}(\nabla\phi)^2 + V(\phi) + \frac{1}{2h^2}(\phi - 2\phi_{n-1} + \phi_{n-2})^2\} \quad n = 2, 3, \cdots \quad ,$$
$$V(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \quad , \quad \phi = \phi(\mathbf{x}) \ , \ \phi_{n-1} = \phi_{n-1}(\mathbf{x}), \ \phi_{n-2} = \phi_{n-2}(\mathbf{x}) \quad , \tag{62}$$

where ϕ_{n-2} and ϕ_{n-1} are given. $(\mathbf{x}) = (x_1, x_2, x_3)$ is the 3 dimensional spacial coordinates. The step n configuration $\phi_n(\mathbf{x})$ is defined to be the energy minimal one.

$$\left.\frac{\delta I_n}{\delta\phi}\right|_{\phi=\phi_n} = -\nabla^2\phi_n + \left.\frac{\delta V}{\delta\phi}\right|_{\phi=\phi_n} + \frac{1}{h^2}(\phi_n - 2\phi_{n-1} + \phi_{n-2}) = 0 \quad ,$$
$$\frac{\delta V}{\delta\phi} = m^2\phi + \frac{\lambda}{3!}\phi^3 \quad . \tag{63}$$

Using the step-time notation: $\phi_n(\mathbf{x}) \equiv \phi(\mathbf{x}, t_n), \ t_n = nh$, we obtain, in the continuous time limit $h \to +0$,

$$(\partial_t^2 - \nabla^2 + m^2)\phi + \frac{\lambda}{3!}\phi^3 = 0 \quad . \tag{64}$$

This is the (3+1) dim ϕ^4 scalar field equation.

10 Appendix B Calculation of Fluctuation Effect

In Sec.4, we have developed the method of calculating the statistical fluctuation occurring in the (1-dim) viscous fluid matter. The background-field

method is taken. At the 1-loop approximation, the key quantity to calculate the energy functional $\Gamma[u(x)]$ is the *heat-kernel* $G(x,y;\tau)$ given by (see eq.(30))

$$< x|\mathrm{e}^{-\tau D}|y > \equiv G(x,y;\tau) \quad ,$$
$$(\frac{\partial}{\partial\tau} + D)G(x,y;\tau) = 0 \quad , \quad \lim_{\tau\to+0} G(x,y;\tau) = \delta(x-y) \quad ,$$
$$D = -\epsilon^{-1}\frac{\partial^2}{\partial x^2} - \bar{V}(x) \quad , \quad \bar{V}(x) = -\lambda u_n(x)^2 - m^2 - \frac{1}{h} - \frac{du_{n-1}}{dx} \quad , \quad (65)$$

where $|x>$ and $<x|$ are Dirac's abstract state vectors. $\epsilon \equiv \tilde{\rho}_0/\sigma$ is explicitly written to show the dimension consistency. In the text, we take $\epsilon = 1$. For the calculation, in this appendix, we change the scale of τ and D as follows.

$$\tau \quad \to \quad \frac{\tau}{\epsilon} = \tilde{\tau} \quad , \quad D \quad \to \quad \epsilon D = \tilde{D} \quad ,$$
$$\tilde{D} = -\frac{\partial^2}{\partial x^2} - \epsilon\bar{V}(x) \quad , \quad [\tilde{\tau}]=\mathrm{L}^2 \quad , \quad [\tilde{D}]=\mathrm{L}^{-2} \quad , \quad [\epsilon]=1/\mathrm{LM} \quad . \quad (66)$$

In the following within this appendix, for simplicity we omit the symbol of 'tilde'.

The kernel is formally solved as

$$G(x,y;\tau) = G_0(x-y;\tau) + \int dz \int_0^\tau d\omega S(x-z;\tau-\omega)\epsilon\bar{V}(z)G(z,y;\omega) \quad ,(67)$$

where $G_0(x-y;\tau)$ and the (heat-)propagator $S(x-y;\tau)$ are given by

$$G_0(x-y;\tau) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\mathrm{e}^{-k^2\tau+ik(x-y)} = \frac{1}{\sqrt{4\pi\tau}}\mathrm{e}^{-\frac{(x-y)^2}{4\tau}} \quad , \quad \tau > 0 \quad ,$$
$$S(x-y;\tau) = \int\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{dk^0}{2\pi}\frac{\exp\{-ik^0\tau+ik(x-y)\}}{-ik^0+k^2} = \theta(\tau)G_0(x-y;\tau) \quad .(68)$$

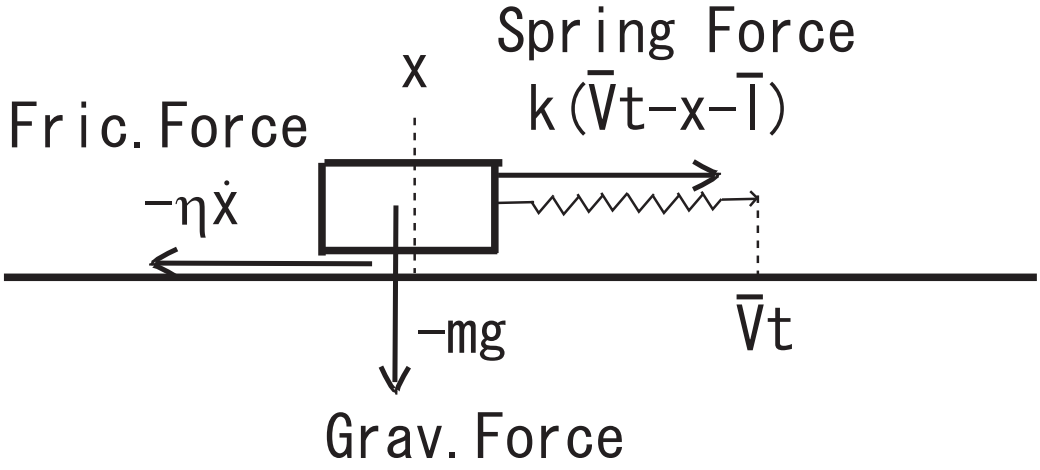
They satisfy

$$(\frac{\partial}{\partial\tau} - \frac{\partial^2}{\partial x^2})G_0(x-y;\tau) = 0 \quad , \quad \tau > 0 \quad , \quad \lim_{\tau\to+0} G_0(x-y;\tau) = \delta(x-y) \quad ,$$
$$(\frac{\partial}{\partial\tau} - \frac{\partial^2}{\partial x^2})S(x-y;\tau) = \delta(\tau)\delta(x-y) \quad , \quad \lim_{\tau\to+0} S(x-y;\tau) = \delta(x-y) \quad .(69)$$

Up to the first order of $\bar{V}$,

$$G(x,x;\tau) = \frac{1}{\sqrt{4\pi\tau}} + \int dz \int_0^\tau d\omega S(x-z;\tau-\omega)\epsilon\bar{V}(z)G_0(z-x;\omega) + O(\bar{V}^2) \quad .(70)$$

Figure 12: The spring-block model, (75).



The second term is evaluated as

$$\int dz \int_0^\tau d\omega G_0(x-z;\tau-\omega)\epsilon\bar{V}(z)G_0(z-x;\omega) =$$
$$\int dz \int_0^\tau d\omega \frac{1}{4\pi}\frac{1}{\sqrt{(\tau-\omega)\omega}}\epsilon\bar{V}(z)\exp\{-\frac{\tau}{4(\tau-\omega)\omega}(x-z)^2\} \quad . \qquad (71)$$

Finally the contribution to $\ln(\det D)^{-1/2} = \frac{1}{2}\int_0^\infty (d\tau/\tau)\mathrm{Tr}G(x,y;\tau) = \frac{1}{2}\int_0^\infty (d\tau/\tau)\int_{-l}^{l} dx G(x,x;\tau)$ is evaluated as

$$\frac{1}{2}\int_0^\infty d\tau \frac{1}{2}\frac{1}{\sqrt{\tau\pi}}\int dz\epsilon\bar{V}(z) = \frac{1}{4\sqrt{\pi}}\int_0^{\epsilon^{-1}\mu^{-1}} \tau^{-1/2}d\tau \int dz\epsilon\bar{V}(z) =$$
$$\frac{l}{\sqrt{\pi}}\sqrt{\epsilon\Lambda} - \frac{1}{2\sqrt{\pi\epsilon\mu}}\int_{-l}^{l} dz\epsilon\left(\lambda u_n(z)^2 + m^2 + \frac{1}{h} + \frac{du_{n-1}(z)}{dz}\right) \quad , \qquad (72)$$

where the infrared cut-off parameter $\mu \equiv \sqrt{\sigma}/l$ and the ultraviolet cut-off parameter $\Lambda \equiv h^{-1}$ are introduced. [28]

11 Appendix C Spring-Block Model

In Sec.6, the movement of the harmonic oscillator with friction was examined. Here we treat the movement of a block which is pulled by the spring which

[28] The dimensions of these parameters are $[\mu]=[\Lambda]=\mathrm{M/L}$. The space-integral part $(\int dx\cdots)$ in (72) is evaluated as $\int_{-l}^{l} dx \exp\{-(\tau/4(\tau-\omega)\omega)(x-z)^2\} \sim \int_{-\infty}^{\infty}\exp\{''\} = 2\sqrt{\pi(\tau-\omega)\omega/\tau}$ where l is safely extended to infinity.

moves at the constant speed $\bar{V}$. The block moves on the surface with friction. We take the following n-th energy function to define the step flow.

$$K_n(x) = V(x) - hnk\bar{V}x + \frac{\eta}{2h}(x - x_{n-1})^2 + \frac{m}{2h^2}(x - 2x_{n-1} + x_{n-2})^2 + K_n^0 \quad ,$$
$$V(x) = \frac{kx^2}{2} + k\bar{l}x \quad ,(73)$$

where η is the friction coefficient and m is the block mass. The potential $V(x)$ has two terms: one is the harmonic oscillator with the spring constant k, and the other is the linear term of x with a new parameter $\bar{l}$ (the natural length of the spring). $\bar{V}$ is the velocity (constant) with which the front-end of the spring moves. K_n^0 is a constant which does not depend on x. It will be fixed later. The n-th step x_n is determined by the energy minimum principle: $\delta K_n(x)|_{x=x_n} = 0$.

$$\frac{k}{m}(x_n + \bar{l} - nh\bar{V}) + + \frac{\eta}{m}\frac{1}{h}(x_n - x_{n-1}) + \frac{1}{h^2}(x_n - 2x_{n-1} + x_{n-2}) = 0 \quad \text{or}$$
$$x_n = \frac{\omega^2(-\bar{l} + nh\bar{V}) + \frac{\eta'}{h}x_{n-1} + \frac{1}{h^2}(2x_{n-1} - x_{n-2})}{\omega^2 + \frac{\eta'}{h} + \frac{1}{h^2}},\ \omega \equiv \sqrt{\frac{k}{m}}\ ,\ \eta' \equiv \frac{\eta}{m} \quad .(74)$$

For the continuous limit: $h \to 0, nh = t_n \to t, (x_n - x_{n-1})/h \to \dot{x}, (x_n - 2x_{n-1} + x_{n-2})/h^2 \to \ddot{x}$, the above recursion relation reduces to the following differential equation. [29]

$$m\ddot{x} = k(\bar{V}t - x - \bar{l}) - \eta\dot{x} \quad . \qquad (75)$$

We keep the step-wise approach. The system energy given by $K_n(x_n)$. Taking the constant term K_n^0 as

$$K_n^0 = -V(x_n) - \frac{m}{2h^2}(x_n - 2x_{n-1} + x_{n-2})^2 + V(x_0) + \frac{m}{2h^2}(x_1 - x_0)^2$$
$$+hnk\bar{V}x_n + hk\bar{V}x_n \quad , \quad (76)$$

the energy is given as

$$K_n(x_n) = hk\bar{V}x_n + \frac{\eta}{2h}(x_n - x_{n-1})^2 + V(x_0) + \frac{m}{2h^2}(x_1 - x_0)^2 \quad , \qquad (77)$$

We have taken the constant term K_n^0, (76), in such a way that the system keeps the constant energy when the energy dissipation does not occur ($\eta = 0, \bar{V} = 0$). The last two terms in (76) comes from the following relation.

$$\frac{d}{dt}\left(\frac{1}{2}k\dot{x}^2 - k\bar{V}xt + V(x)\right) = -\eta\dot{x}^2 - k\bar{V}x \quad . \qquad (78)$$

[29] This equation is called spring-block model and is used to explain some aspect (stick-slip motion, etc) of the earthquake.

Note that the $\bar{V}$-terms appear both hand-sides, the total derivative terms and the dissipative terms. [30]

The graphs of movement (x_n, eq.(74)) and energy change ($K_n(x_n)$, eq.(77)) are shown in Fig.13 and Fig.14 respectively. From the graph of Fig.14, we see this system reaches the steady energy-state as $n \to \infty$, (24).

12 Acknowledgment

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[30] The role of last 2 terms of (76) are as follows. The first one ($+hnk\bar{V}x_n$) is for canceling the term $-hnk\bar{V}x$ in $K_n(x)$, (73). The second one ($+hk\bar{V}x_n$) is for taking care of the second dissipative term of the RHS of (78), $-k\bar{V}x$.

Figure 13: Spring-Block Model, Movement, h=0.0001,$\sqrt{k/m}$=10.0, η/m=1.0, $\bar{V}$=1.0, $\bar{l}$=1.0, total step no =20000. The step-wise solution (74) correctly reproduces the analytic solution: $x(t) = \mathrm{e}^{-\eta' t/2}\bar{V}\{(\eta'^2/2\omega^2 - 1)(\sin\Omega t)/\Omega + (\eta'/\omega^2)\cos\Omega t\} - \bar{l} + \bar{V}(t - \eta'/\omega^2)$,$\Omega = (1/2)\sqrt{4\omega^2 - \eta'^2} = 9.99$, $0 \leq t \leq 2$, $x(0) = -\bar{l},\ \dot{x}(0) = 0$.

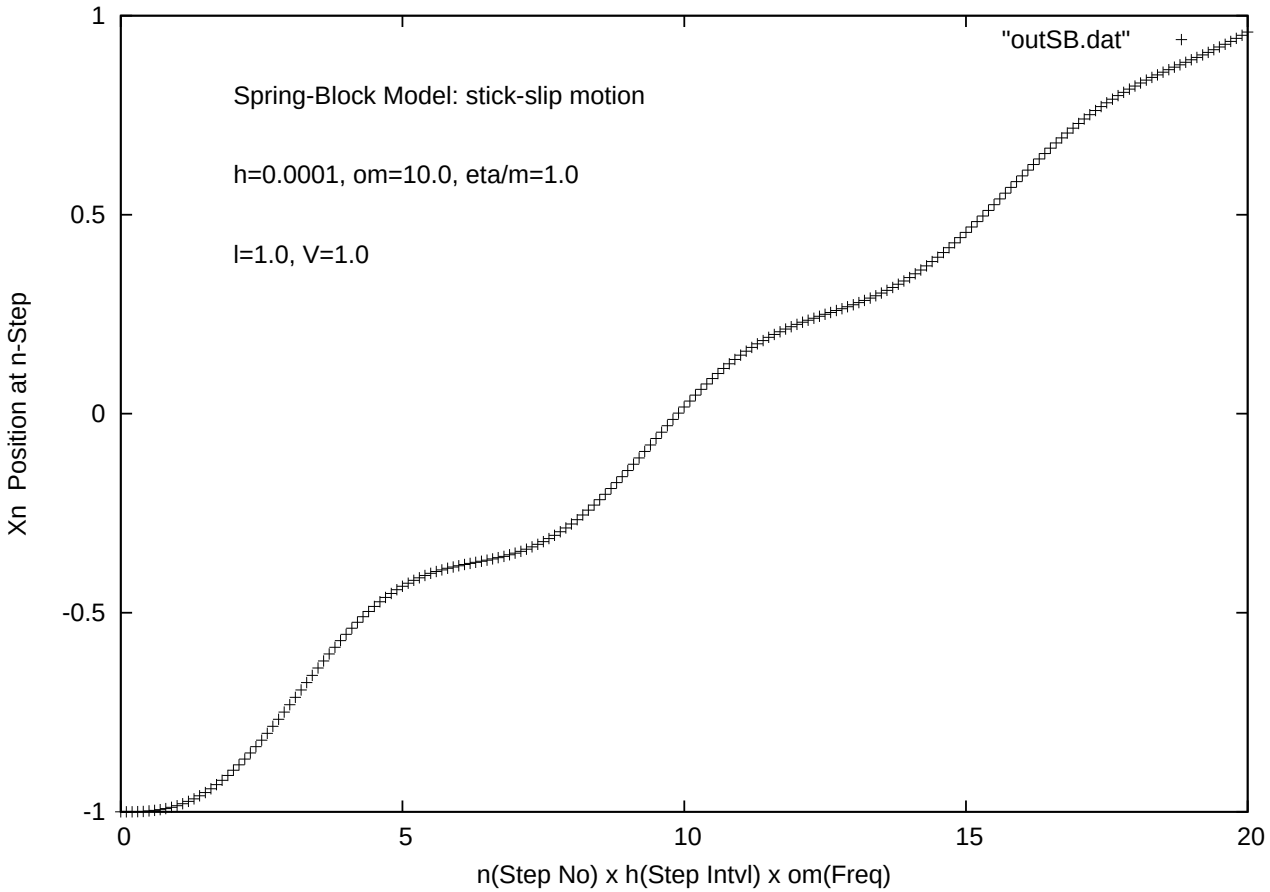
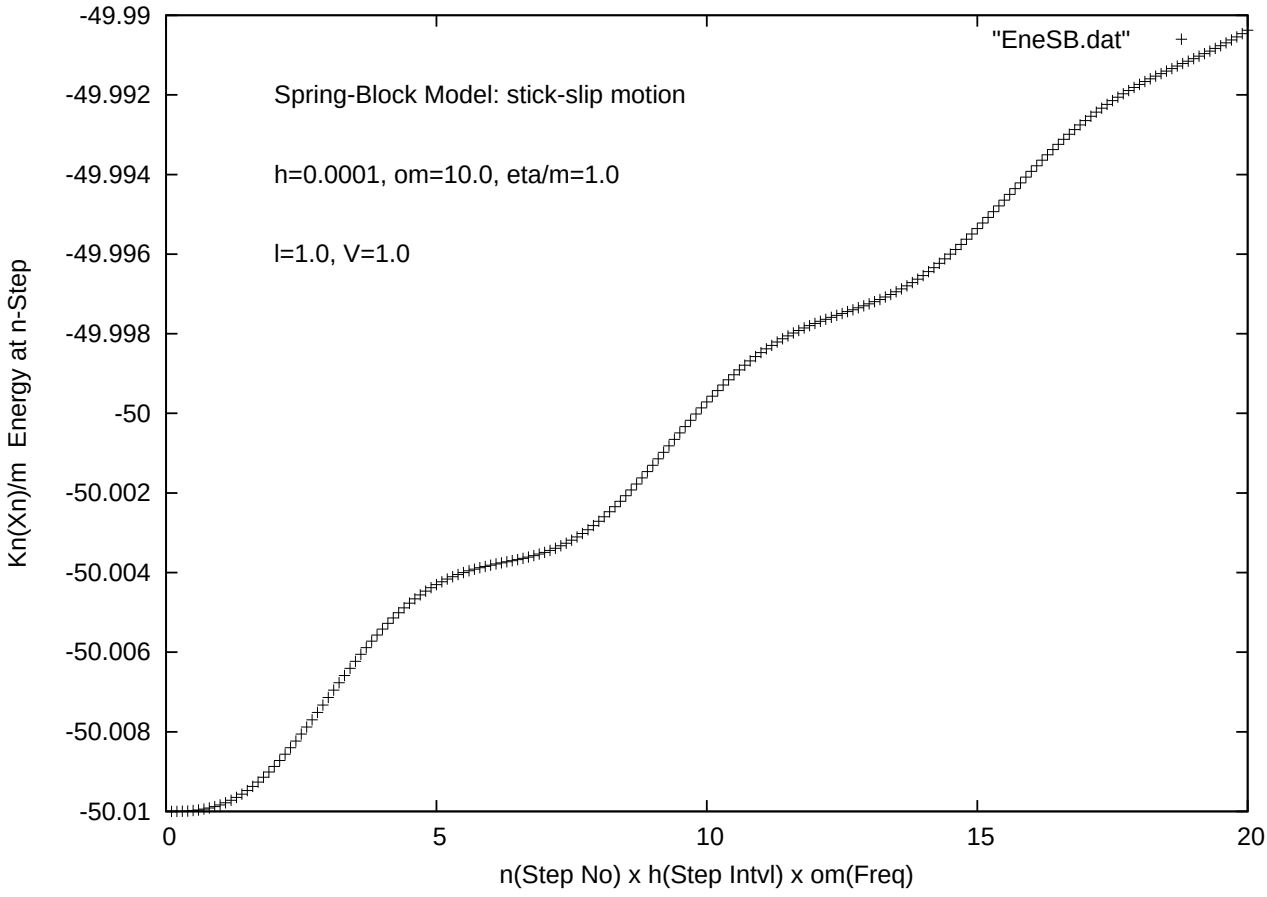


Figure 14: Spring-Block Model, Energy Change, h=0.0001,$\sqrt{k/m}$=10.0, η/m=1.0, $\bar{V}$=1.0, $\bar{l}$=1.0, total step no =20000.



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